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Graph Minors. VII. Disjoint Paths on a Surface

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Let s_1, t_1, s_2, t_2, ..., s_k, t_k be vertices of a graph G drawn in a surface Σ. When are there k vertex-disjoint paths of G linking s_i and t_i $(1 \leqslant i \leqslant k)$? We study sufficient conditions—for instance, it suffices that G is connected and "uses up" the surface adequately, and all the s_i's and t_j's are mutually "far apart." Our results are applied to yield a polynomially bounded algorithm to solve the problem for fixed Σ and k.

1. INTRODUCTION

Let G be a graph drawn on a connected surface Σ, and let s_1, t_1, s_2, t_2, ..., s_k, t_k be vertices of G. When are there k vertex-disjoint paths joining s_i and t_i $(1 \leqslant i \leqslant k)$, respectively? It is plausible that if the vertices s_1, t_1, ..., s_k, t_k are in some sense far apart and if G "represents" the surface Σ adequately, the paths will exist. More precisely, if

(i) G is connected,

(ii) every curve drawn in Σ between two distinct members of $\{s_1, t_1, ..., s_k, t_k\}$ has large "length", that is, has a large number of points in common with the drawing of G,

(iii) every closed curve drawn in Σ which is not null-homotopic also has large "length," and

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(iv) every separating closed curve drawn in Σ which separates $\{s_1, t_1, ..., s_k, t_k\}$ into two sets each with at least two members has large "length,"

then the paths will exist. This is indeed true, and is a consequence of our main result, which is more complicated but of a similar type. The new complication is introduced because we find that this kind of sufficient condition for the existence of the paths is much more useful if we relax condition (ii)—instead of asking that $s_1, t_1, ..., s_k, t_k$ be pairwise far apart, we ask that they fall into groups, each group lying on the boundary of one region and distinct groups being far apart.

Its applications are as follows:

(i) We obtain a polynomially bounded algorithm, for any fixed integer k and fixed surface Σ, to determine if the paths exist. The idea is basically that we test if our theorem can be applied. If so, the paths exist. If not, then either the paths clearly do not exist, or we find an offending curve which is too short (which we can choose so that it passes only through vertices and regions of G), and "split" the vertices it passes through. We can translate our original problem into a set of problems on this new simpler graph.

(ii) A graph is a *minor* of another if it can be obtained from a subgraph of the second by contraction. We prove that for every graph H which can be drawn in a connected surface Σ, not a sphere, there is a number w with the following property. Every graph G drawn in Σ which has no non-null-homotopic closed curve of "length" $\leqslant w$ has a minor isomorphic to H.

(iii) In the next paper of this series [5], we shall use the result of this paper to show that if $G_1, G_2, ...,$ is an infinite sequence of graphs which can be drawn in Σ, there exist $j > i \geqslant 1$ such that G_i is isomorphic to a minor of G_j. In particular, this implies that for every surface Σ there is a "Kuratowski-type" theorem; there is a finite list of graphs $G_1, ..., G_k$ such that an arbitrary graph can be drawn in Σ if and only if it topologically contains none of $G_1, ..., G_k$. In addition, our present result will have numerous applications in later papers of the series, which we do not detail here.

2. Surfaces and Nets

We begin by establishing some terminology and preliminary lemmas. A *surface* is a compact 2-manifold with (possibly null) boundary. The boundary of a surface Σ is denoted by $bd(\Sigma)$, and each component of $bd(\Sigma)$ is called a *cuff*. (Every cuff is homeomorphic to the circle S^1.) $\Sigma(a, b, c)$

denotes the surface obtained from a 2-sphere by adding a handles and b cross-caps, and then removing the interiors of c pairwise disjoint closed discs. Thus $\Sigma(0, 0, 1)$ is a closed disc, $\Sigma(0, 0, 2)$ is a cylinder, $\Sigma(1, 0, 0)$ is a torus, $\Sigma(0, 1, 0)$ is a projective plane, and $\Sigma(0, 1, 1)$ is a Möbius band. It is known that every connected surface is homeomorphic to $\Sigma(a, b, c)$ for some choice of a, b, c. We denote "Σ is homeomorphic to Σ'" by $\Sigma \cong \Sigma'$. We denote the closure of any $X \subseteq \Sigma$ by $\bar{X}$.

A subset $F \subseteq \Sigma$ homeomorphic to S^1 is called an *O-arc*. An *I-arc* is a subset $F \subseteq \Sigma$ homeomorphic to $[0, 1]$, such that there is a homeomorphism $f\colon [0, 1] \to F$ with $f(0)$, $f(1) \in bd(\Sigma)$. In either case a homeomorphism from S^1 (respectively, $[0, 1]$) to F is called an *arc-map*. If F is an I-arc with arc-map f, we call $f(0)$ and $f(1)$ the *ends* of F.

An O-arc F is *null-homotopic* in Σ if there is a homotopy of some arc-map onto a constant map. Then by a result of [2], F is null-homotopic if and only if there is a closed disc $\Delta \subseteq \Sigma$ with $bd(\Delta) = F$, or again if and only if $F \subseteq \Delta$ for some closed disc $\Delta \subseteq \Sigma$.

In this paper we are concerned with graphs drawn on surfaces, and we formulate a non-standard definition of "graph" to avoid continually having to refer to drawings. A *graph* G then is a pair $(U(G), V(G))$, where $U(G)$ is a topological space and $V(G) \subseteq U(G)$ is finite, such that

(i) $U(G) - V(G)$ has only a finite number of components (called *edges*), and

(ii) if e is an edge then $(e, \bar{e})$ is homeomorphic either to $((0, 1), [0, 1])$ or to $(S^1 - \{x\}, S^1)$, where $x \in S^1$.

$V(G)$ is called the *vertex set* of G and its members are called *vertices*. If Σ is a surface, a *graph in* Σ means a graph $G = (U(G), V(G))$ where $U(G)$ is a subspace of Σ. If G is a graph in Σ, a subset $X \subseteq \Sigma$ is *G-normal* if $X \cap e = \emptyset$ for each edge e of G. A graph G in Σ is *proper* if $bd(\Sigma)$ is G-normal. A component of $\Sigma - U(G)$ is called a *region* of G. The rest of our basic graph-theoretic terminology is more or less standard. We mention that *paths* and *circuits* have no "repeated" vertices.

If J is a graph in Σ with $bd(\Sigma) \subseteq U(J)$, then for every edge e of J, either $e \subseteq bd(\Sigma)$ or $e \cap bd(\Sigma) = \emptyset$. We denote by $J - bd(\Sigma)$ the subgraph $(V(J) \cup (U(J) - bd(\Sigma)), V(J))$ of such a graph J. Evidently $J - bd(\Sigma)$ is proper in Σ.

An I-arc F is a *boundary I-arc* if $F \subseteq bd(\Sigma)$. Two I-arcs F_1, F_2 are *internally disjoint* if every point in $F_1 \cap F_2$ is an end of both of them. Take two internally disjoint boundary I-arcs in Σ, with arc-maps f, g, respectively. If we make the identifications $f(x) = g(x) (0 \leqslant x \leqslant 1)$ we obtain a new surface. We call this operation *pasting* $f = g$, and the inverse operation *cutting* (*along* the appropriate subset of the surface). More generally, for any proper graph J drawn in a surface Σ with no isolated vertices, we can "cut" along $U(J)$ in the obvious way, and obtain a new surface Σ'. There is a natural surjection $\phi: \Sigma' \to \Sigma$, which we call the *associated surjection.* For $z \in \Sigma$, $\phi^{-1}(z)$ denotes $\{z' \in \Sigma' : \phi(z') = z\}$, and for $Z \subseteq \Sigma$, $\phi^{-1}(Z)$ denotes $\bigcup (\phi^{-1}(z): z \in Z)$. If G is a proper graph in Σ' and $U(J)$ is G-normal, then $(\phi^{-1}(U(G)), \phi^{-1}(V(G)))$ is a proper graph in Σ', which we denote by $\phi^{-1}(G)$.

It is known that any connected surface can be constructed by repeated pasting, starting from a closed disc. A *net* (Δ, Π) for a connected surface Σ is a closed disc Δ together with a set

$$\Pi = \{\{f_1, g_1\}, ..., \{f_r, g_r\}\},$$

where $f_1, g_1, ..., f_r, g_r$ are arc-maps of pairwise internally disjoint boundary I-arcs in Δ, such that Σ can be obtained by pasting $f_1 = g_1$, $f_2 = g_2, ..., f_r = g_r$. Let $\phi: \Delta \to \Sigma$ be the associated surjection. Let $J = (U, V)$, where $U = \{\phi(x): x \in bd(\Delta)\}$ and

$$V = \{z \in \Sigma : \text{for some } \{f, g\} \in \Pi, f \text{ has an end in } \phi^{-1}(z)\}.$$

Then, provided that $\Pi \neq \emptyset$, J is a graph which we call the *seam graph* for the net (Δ, Π).

(2.1) *Let J be a graph in Σ. Then J is a seam graph for some net if and only if*

(i) *$bd(\Sigma) \subseteq U(J)$, and for every edge e of J, either $e \subseteq bd(\Sigma)$ or $e \cap bd(\Sigma) = \emptyset$,*

(ii) *$J - bd(\Sigma)$ has no isolated vertices,*

(iii) *J has a unique region, and*

(iv) *every O-arc included in that region is null-homotopic.*

Proof. Suppose that J is a seam graph for some net (Δ, Π), where $\Pi = \{\{f_1, g_1\}, ..., \{f_r, g_r\}\}$. Let ϕ be the associated surjection. Then the restriction of ϕ to $\Delta - bd(\Delta)$ is injective, and hence provides a homeomorphism from $\Delta - bd(\Delta)$ to $\Sigma - U(J)$. Since $\Delta - bd(\Delta)$ is connected, it follows that $\Sigma - U(J)$ is connected, that is, J has a unique region; and since every O-arc included in $\Delta - bd(\Delta)$ is null-homotopic, the same is true for $\Sigma - U(J)$. Thus (iii) and (iv) are verified. To verify (i) and (ii), we observe that the sets

$$\{f_i(x): 0 < x < 1\} \qquad (1 \leqslant i \leqslant r)$$

are all edges of J, and each is disjoint from $bd(\Sigma)$; and any other edge of J is included in $bd(\Sigma)$. Then (i) and (ii) follow.

For the converse, suppose J satisfies (i)–(iv). Let Δ be the surface obtained from Σ by cutting along $U(J-bd(\Sigma))$. Then by (iii), $\Delta - bd(\Delta)$ is connected, and by (iv) every O-arc included in $\Delta - bd(\Delta)$ is null-homotopic. Since Δ is a surface, it follows that Δ is a closed disc, as required.

(2.2) *Let $\Sigma \cong \Sigma(a, b, c)$ and let J be a seam graph in Σ for some net. Suppose that every vertex of J has valency at least* 3. *Then* $|V(J)| \leqslant 2(2a+b+c-1)$, *and* $|E(J)| \leqslant 3(2a+b+c-1)$.

Proof. Let Σ' be the surface obtained from Σ by pasting a closed disc onto each cuff of Σ. Then $\Sigma' \cong \Sigma(a, b, 0)$. Now J is a graph in Σ' with $c+1$ regions, and they are all simply connected (for definition, see Section 12). We may apply Euler's formula to deduce that

$$|V(J)| - |E(J)| + c + 1 = 2 - 2a - b.$$

But every vertex of J in Σ' has valency $\geqslant 3$, and so $|V(J)| \leqslant \frac{2}{3}|E(J)|$. The result follows.

Let (Δ, Π) be a net for Σ, and let G be a proper graph in Σ. We say that (Δ, Π) is *G-normal* if $U(J)$ is G-normal, where J is the seam graph for (Δ, Π). It is easy to see that if Σ is connected and G is a proper graph in Σ, there is a G-normal net. (We sketch a proof. It suffices to show this when G is connected and every region of G intersects $\Sigma - bd(\Sigma)$ in an open disc, for we can always augment G to make this true. For each region r choose a point $v_r \in r - bd(\Sigma)$, and for each vertex v incident with r make a cut in $(r - bd(\Sigma)) \cup \{v\}$ from v_r to v, in such a way that all these cuts are internally disjoint, in the natural sense. We obtain a surface, each component of which is a disc. Now paste back together just enough of these cuts to make the surface connected; the result is the required net.)

Let G be a proper graph in a connected surface Σ. A net (Δ, Π) with associated surjection $\phi: \Delta \to \Sigma$ is *minimal* (with respect to G) if

(i) it is G-normal,

(ii) among all G-normal nets, (Δ, Π) has $|\phi^{-1}(V(G)) \cap bd(\Delta)|$ minimum, and

(iii) among all G-normal nets satisfying (ii), (Δ, Π) has $|\Pi|$ minimum.

We must establish some properties of minimal nets.

(2.3) *Let Σ be a connected surface with $\Sigma \not\cong \Sigma(0,0,0)$, $\Sigma(0,0,1)$,*

$\Sigma(0, 1, 0)$. *Let J be the seam graph for some minimal net with respect to some proper graph G in Σ. Then no vertex of J has valency $\leqslant 2$.*

Proof. Certainly J has no isolated vertices. If v is a vertex of valency 1, let J' be the graph obtained from J by deleting v. Then since $\Sigma \not\cong \Sigma(0, 0, 0)$, J' is the seam graph for a net which contradicts the minimality of our original net. Now suppose v has valency 2. No loop of J is incident with v since $\Sigma \not\cong \Sigma(0, 0, 1)$, $\Sigma(0, 1, 0)$; and so $(U(J), V(J) - \{v\})$ is a graph which again is a seam graph for a smaller net. This completes the proof.

We deduce from (2.3) and (2.2) that

(2.4) *If $\Sigma \cong \Sigma(a, b, c)$ where $2a + b + c > 1$, and J is the seam graph for some minimal net for Σ (with respect to some G), then $|V(J)| \leqslant 2(2a + b + c - 1)$ and $|E(J)| \leqslant 3(2a + b + c - 1)$.*

Let (Δ, Π) be a net for Σ. Let J be the associated seam graph, and let $\phi: \Delta \to \Sigma$ be the associated surjection. Let $s, t \in bd(\Delta)$ be distinct, and let C_1, C_2 be the two components of $bd(\Delta) - \{s, t\}$. Let

$$J' = (U(J), V(J) \cup \{\phi(s), \phi(t)\}).$$

The edges of J' are of four types:

(i) edges e with $e \subseteq bd(\Sigma)$,

(ii) edges e such that for all $z \in e$, $\phi^{-1}(z) \cap C_i \neq \emptyset (i = 1, 2)$,

(iii) edges e such that $e \cap bd(\Sigma) = \emptyset$ and for all $z \in e$, $\phi^{-1}(z) \subseteq C_1$,

and

(iv) edges e such that $e \cap bd(\Sigma) = \emptyset$ and for all $z \in e$, $\phi^{-1}(z) \subseteq C_2$.

(This follows easily from the definition of a net.)

(2.5) *There is a path of J' between $\phi(s)$ and $\phi(t)$ with no edges of type* (iii) *or* (iv).

Proof. The image of $\bar{C}_1$ under ϕ yields a sequence $\phi(s) = v_0$, e_1, v_1, $e_2, ..., e_k$, $v_k = \phi(t)$ of vertices and edges of J', such that for $1 \leqslant i \leqslant k$, e_i has ends v_{i-1} and v_i. Each edge of J' of type (iii) occurs twice in this sequence, while those of type (iv) do not occur. Let J'' be the subgraph of J' consisting of those edges which occur exactly once in the sequence, with $V(J'') = V(J')$. By counting we find that every vertex of J'' distinct from $\phi(s)$ and $\phi(t)$ has even valency in J'', while if $\phi(s) \neq \phi(t)$ they both have odd valency. It follows that $\phi(s)$, $\phi(t)$ are in the same component of J'', and the theorem is true.

An I-arc F in Σ is *proper* if $|F \cap bd(\Sigma)| = 2$.

(2.6) *Let (Δ, Π) be a minimal net with respect to a proper graph G in Σ, where $\Sigma \cong \Sigma(a, b, c)$ and $2a+b+c>1$. Let J be the associated seam graph and let ϕ be the associated surjection. Let F be a $\phi^{-1}(G)$-normal proper I-arc in Δ with ends s, t, and let $F^* = F - \{s, t\}$. Then there is a G-normal O-arc A in Σ with*

$$|A \cap V(G)| \leqslant 2(2a+b+c)(|\phi(F^*) \cap V(G)| + 1) + |bd(\Sigma) \cap V(G)|$$

and with $\phi(F) \subseteq A \subseteq \phi(F) \cup U(J)$.

Proof. As in (2.5), let J' be $(U(J), V(J) \cup \{\phi(s), \phi(t)\})$. Let e be any edge of J' of type (ii). Then there exists $\{f, g\} \in \Pi$ such that

$$e = \{\phi(f(x)) : 0 < x < 1\} = \{\phi(g(x)) : 0 < x < 1\}.$$

Let Δ^* be the surface obtained from Δ by pasting $f = g$ and cutting along F. Then Δ^* is a closed disc since F is proper and e is of type (ii), and the new net we obtain is also G-normal. Since (Δ, Π) is a minimal net, we have

$$|e \cap V(G)| \leqslant |\phi(F^*) \cap V(G)|.$$

Let P be the path of J' provided by (2.5). For each edge e of type (ii) of J', the inequality above holds, while if e is an edge of J' of type (i) then $\bar{e} \cap V(G) \subseteq bd(\Sigma)$. It follows that

$$|U(P) \cap V(G)| \leqslant |E(P)| \cdot |\phi(F^*) \cap V(G)| + |V(P)| + |bd(\Sigma) \cap V(G)|.$$

Combining P with $\phi(F^*)$ yields the required G-normal O-arc, since

$$|E(P)| \leqslant |V(P)| \leqslant |V(J')| \leqslant |V(J)| + 2 \leqslant 2(2a+b+c)$$

by (2.4).

3. Matchings and Forests

Let Σ be a surface and let F, F' be proper I-arcs in Σ. We say that F is *similar* to F' if there is a homeomorphism $\alpha: \Sigma \to \Sigma$ such that $\alpha(F) = F'$. This is an equivalence relation, and we call the equivalence classes *similarity classes*.

(3.1) *For any fixed Σ there are only finitely many similarity classes.*

Proof. Let F be a proper I-arc in Σ. If we cut along F we obtain a new surface Σ^* and two boundary I-arcs X, Y say in Σ^* with arc-maps f, g, respectively, such that Σ is obtained from Σ^* by pasting $f=g$. There are, up to homeomorphism, only finitely many possibilities for Σ^* and hence for $(\Sigma^*, X, Y, f(0), g(0))$. But the similarity class of F is determined by the homeomorphism class of $(\Sigma^*, X, Y, f(0), g(0))$ and the result follows.

Now let Σ be a surface. Two proper I-arcs F, F', are *parallel* if there is a homeomorphism $\alpha: \Sigma \to \Sigma$ such that $\alpha(F)=F'$ and $\alpha(x)=x$ for all $x \in bd(\Sigma)$. The equivalence classes of this equivalence relation we call *parallel classes.* We observe that if F, F' are parallel then they are similar.

(3.2) *For any fixed Σ and s, $t \in bd(\Sigma)$, there are only finitely many parallel classes of I-arcs with ends s, t.*

Proof. Let $\Sigma \cong \Sigma(a, b, c)$. By (3.1) it suffices to show that if $\mathscr{F}$ is a set of mutually similar proper I-arcs in Σ, each with ends s, t, then $\mathscr{F}$ is divided by parallelness into only finitely many classes. Let us assign an orientation to each cuff of Σ. For each homeomorphism $\alpha: \Sigma \to \Sigma$, its *signature* is the function σ_α mapping each cuff C to $(\alpha(C), \pm 1)$, where we choose $+1$ if the orientation of C is mapped under α to the orientation of the cuff $\alpha(C)$, and -1 otherwise. The signature is *null* if for each cuff C, $\sigma_\alpha(C)=(C, 1)$. We observe

(1) *Let F, $F' \in \mathscr{F}$. If there is a homeomorphism $\alpha: \Sigma \to \Sigma$ with $\alpha(F)=F'$ and with null signature, then F is parallel to F'.*

Choose $F^* \in \mathscr{F}$. For each $F \in \mathscr{F}$ there exists a homeomorphism $\alpha_F: \Sigma \to \Sigma$ with $\alpha_F(F^*)=F$. Now for F, $F' \in \mathscr{F}$, if α_F and $\alpha_{F'}$ have the same signature, then F is parallel to F' by (1), for $\alpha=\alpha_{F'} \cdot \alpha_F^{-1}$ satisfies the hypothesis of (1). Yet there are only finitely many possible signatures, and the result follows.

A *matching* in a surface Σ is a proper graph G in Σ with $V(G) \subseteq bd(\Sigma)$, in which every vertex has valency 1. For $X \subseteq bd(\Sigma)$, an *X-matching* is a matching G with $V(G)=X$. Two X-matchings G, G' in Σ are said to be *congruent* if there is a homeomorphism $\alpha: \Sigma \to \Sigma$ such that $\alpha(U(G))=U(G')$ and $\alpha(x)=x$ for all $x \in bd(\Sigma)$. We call the equivalence classes of this equivalence relation *congruence classes.*

(3.3) *For any surface Σ and finite $X \subseteq bd(\Sigma)$, there are only finitely many congruence classes of X-matchings.*

Proof. We proceed by induction on $|X|$. There are only finitely many possibilities for the pairing of the members of X given by the edges of the matching. Having fixed that pairing, there are by (3.2) essentially only finitely many choices for the first edge of the matching. Having fixed that edge we may cut along it, and the result follows from our inductive hypothesis.

A *forest* in a surface Σ is a proper graph in Σ with no circuits. Two forests H_1, H_2 are *homotopic* in Σ if

(i) $V(H_1) \cap bd(\Sigma) = V(H_2) \cap bd(\Sigma)$,

(ii) for $s, t \in V(H_1) \cap bd(\Sigma)$, there is a path of H_1 from s to t if and only if there is such a path in H_2, and

(iii) for $s, t \in V(H_1) \cap bd(\Sigma)$, if P_i is a path of H_i from s to t ($i = 1, 2$) then P_1 is homotopic in Σ to P_2.

(Homotopy of paths in Σ is defined in the usual way.) We say that forests H_1, H_2 in Σ are *homoplastic* if there is a homeomorphism $\alpha: \Sigma \to \Sigma$ such that

(i) $\alpha(x) = x$ for all $x \in bd(\Sigma)$, and

(ii) the forest $\alpha(H_1)$ is homotopic in Σ to H_2.

The equivalence classes of this equivalence relation are called *homoplasty classes*. For $Y \subseteq bd(\Sigma)$, a *Y-forest* is a forest H with $V(H) \cap bd(\Sigma) = Y$.

(3.4) *For any surface Σ and finite $Y \subseteq bd(\Sigma)$, there are only finitely many homoplasty classes of Y-forests.*

Proof. For each $y \in Y$, choose a boundary I-arc F_y with $y \in F_y$, such that y is not an end of F_y, and $F_y \cap F_{y'} = \emptyset$ for distinct $y, y' \in Y$. Let X be the set of ends of all the F_y ($y \in Y$); thus $|X| = 2\,|Y|$. An X-matching G *surrounds* a Y-forest H if $U(G) \cap U(H) = \emptyset$ and for each component H' of H with $V(H') \cap bd(\Sigma) \neq \emptyset$ there is a simply connected region of G including $U(H')$. We observe

(1) *Every Y-forest is surrounded by some X-matching.*

For if H is a Y-forest, let us "thicken" each edge of H slightly, so that each component of H is enlarged into a closed disc, the discs are mutually disjoint, and their union intersects $bd(\Sigma)$ in precisely $\bigcup (F_y : y \in Y)$. Then the desired X-matching may be found in the boundary of these discs.

Clearly,

(2) *If G is an X-matching surrounding Y-forests H, H′ then H and H′ are homotopic.*

From (2) we deduce

(3) *If H, H′ are Y-forests surrounded by X-matchings G, G′, respectively, and G and G′ are congruent, then H and H′ are homoplastic.*

But from (1), (3), and (3.3), the result follows.

(3.5) *For every matching G in Σ and every homoplasty class $\mathscr{B}$ there exists $H \in \mathscr{B}$ with $U(G) \cap U(H)$ finite.*

The proof is left to the reader.

(3.6) *For every surface Σ and all integers k, $n \geqslant 0$ there is an integer $w(\Sigma, k, n)$ such that for every matching G in Σ with $|V(G)| \leqslant 2n$ and every forest H in Σ with $|V(H) \cap bd(\Sigma)| \leqslant k$, there is a forest H' homoplastic to H with $|U(G) \cap U(H')| \leqslant w(\Sigma, k, n)$.*

Proof. Since $|V(G)| \leqslant 2n$ and $|V(H) \cap bd(\Sigma)| \leqslant k$, there are only finitely many ways to choose $V(G)$ and $V(H) \cap bd(\Sigma)$, up to homeomorphism of Σ. Thus it suffices to show that for finite X, $Y \subseteq bd(\Sigma)$, there is an integer w such that for every X-matching G and Y-forest H, there is a Y-forest H' homoplastic to H with $|U(G) \cap U(H')| \leqslant w$. By (3.3) and (3.4), it suffices to show that for every congruence class $\mathscr{A}$ of X-matchings and every homoplasty class $\mathscr{B}$ of Y-forests there is an integer w such that for every $G \in \mathscr{A}$ there exists $H \in \mathscr{B}$ with $|U(G) \cap U(H)| \leqslant w$. Choose $G_0 \in \mathscr{A}$ and $H_0 \in \mathscr{B}$ with $U(G_0) \cap U(H_0)$ finite (this is possible by (3.5)), and let $w = |U(G_0) \cap U(H_0)|$. We claim that w satisfies our requirements. For let $G \in \mathscr{A}$, and let $\alpha: \Sigma \to \Sigma$ be a homeomorphism with $\alpha(U(G_0)) = U(G)$ and $\alpha(x) = x$ ($x \in bd(\Sigma)$). Let $H = \alpha(H_0)$; then $H \in \mathscr{B}$, and $|U(G) \cap U(H)| = w$, as required.

4. The Main Result–First Version

Let G be a graph in Σ, and let H be a forest in Σ. If there exists a forest H' homoplastic to H which is a subgraph of G, we say that H is *G-feasible.* We are concerned with sufficient conditions for G-feasibility.

Now we apply our results about minimal nets (in particular (2.6)) to prove a preliminary form of our main theorem. We need the following lemma, which is Theorem (3.6) of [4], stated in different language.

(4.1) *Let G be a proper graph in a closed disc Δ and let H be a forest in Δ with $V(H) \cap bd(\Delta) = V(G) \cap bd(\Delta)$. Then H is G-feasible if and only if for every G-normal proper I-arc F with ends s, t say, $|V(G) \cap (F - \{s, t\})| \geqslant |A|$ where C_1, C_2 are the two components of $bd(\Delta) - \{s, t\}$, and A is the set of vertex sets of the components of H which intersect both C_1 and C_2, and contain neither s nor t.*

We shall also need the following four lemmas.

(4.2) *Let Σ be a connected surface with $bd(\Sigma) = \emptyset$, and let $Y \subseteq \Sigma$ be connected and open. Suppose that every O-arc included in Y bounds a disc included in Y. Then either Σ is a sphere and $Y = \Sigma$, or Y is an open disc.*

We sketch a proof. We may assume that $Y \neq \Sigma$. Let Σ' be a universal covering space for Σ, and $\phi: \Sigma' \to \Sigma$ be a covering map (see [1]). Then Σ' is either a sphere or a plane, and so the Riemann mapping theorem holds in Σ'. Thus each component of $\phi^{-1}(Y)$ is an open disc, and the result follows.

(4.3) *Let Σ be a connected surface with at least two cuffs, and let G be a proper graph in Σ. Let $v \geqslant 0$ be an integer, and suppose that every G-normal O-arc F with $F \cap bd(\Sigma) = \emptyset$ and $|V(G) \cap F| \leqslant v$ is null-homotopic. Let C be a cuff. Then there are at least $v+1$ paths of G, each with one end in C and the other in $bd(\Sigma) - C$, mutually vertex-disjoint except for their ends.*

Proof. Let $X \subseteq V(G)$ with $|X| \leqslant v$ and with $X \cap bd(\Sigma) = \emptyset$. By a form of Menger's theorem, it suffices to show that there is a path of G with one end in C and the other in $bd(\Sigma) - C$, and with no vertex in X. Let

$$Z = (\Sigma - (U(G) \cup bd(\Sigma))) \cup X.$$

Then by hypothesis (since $|X| \leqslant v$) every O-arc of Σ included in Z is null-homotopic. Choose Y maximal such that

(i) $Z \subseteq Y$ (and so $r - bd(\Sigma) \subseteq Y$ for every region r of G),

(ii) Y is a union of sets of the form $r - bd(\Sigma)$ (where r is a region of G), e (where e is an edge), and $\{v\}$(where v is a vertex of G not in $bd(\Sigma)$), and

(iii) every O-arc included in Y is null-homotopic.

Then $Y \cap bd(\Sigma) = \emptyset$, from (ii). We claim first that Y is connected. For if not, then from (i) there are regions r_1, r_2 of G, both incident with some edge e, such that $r_1 - bd(\Sigma)$, $r_2 - bd(\Sigma)$ are in different components of Y. But then $Y \cup e$ satisfies (i)–(iii) contrary to the maximality of Y. This proves our claim that Y is connected.

Second, from the maximality of Y, it is easy to see that if $v \in V(G) \cap Y$ then $e \subseteq Y$ for every edge e incident with v. We deduce from (i) and (ii) that there is a subgraph K of G such that

$$Y = \Sigma - (U(K) \cup bd(\Sigma)).$$

Third, let $F \subseteq Y$ be an O-arc. Then F is null-homotopic in Σ, by (iii), and so there is a closed disc $\Delta \subseteq \Sigma$ with $bd(\Delta) = F$. But then $Y \cup \Delta$ satisfies (i)–(iii) (by (11.3)), and so $\Delta \subseteq Y$ from the maximality of Y.

We have shown then that Y is connected and every O-arc included in Y bounds a closed disc included in Y. Moreover, Y is open and $Y \cap bd(\Sigma) = \emptyset$; and so Y is open in the surface obtained from Σ by pasting a closed disc onto each cuff of Σ. By (4.2), either Y is a sphere or Y is an open disc, and so $bd(Y)$ is connected. We deduce that $U(K) \cup bd(\Sigma)$ is connected, and hence there is a path P of K from C to $bd(\Sigma) - C$. But $V(K) \cap X = \emptyset$, since $X \subseteq Y$, and so $V(P) \cap X = \emptyset$. This completes the proof.

(4.4) *Let L be a proper graph in Σ with no isolated vertices, such that no $F \subseteq U(L)$ is an O-arc or an I-arc. Let Σ' be the surface obtained from Σ by cutting along $U(L)$, and let $\phi: \Sigma' \to \Sigma$ be the associated surjection. Let H_0 be a forest in Σ, and for each $z \in V(H_0) \cap bd(\Sigma)$, let $l(z) \in \phi^{-1}(z)$. Then there is a forest H in Σ, homotopic to H_0, such that $U(H) \cap U(L) \subseteq bd(\Sigma)$, and for each $z \in V(H_0) \cap bd(\Sigma)$ every member of $\phi^{-1}(z) - \{l(z)\}$ is an isolated vertex of $\phi^{-1}(H)$.*

The proof is easy (for example, by induction on $|E(L)|$; we remark that if $E(L) \neq \emptyset$ then some vertex of L has valency 1 and is not in $bd(\Sigma)$) and is left to the reader.

(4.5) *Let J be a proper graph in Σ with no isolated vertices. Let Σ' be the surface obtained from Σ by cutting along $U(J)$ and let $\phi: \Sigma' \to \Sigma$ be the associated surjection. Let H be a forest in Σ such that $U(J)$ is H-normal. Then $\phi^{-1}(H)$ is a forest in Σ'. Moreover, if H' is a forest in Σ' homoplastic to $\phi^{-1}(H)$, then $\phi(H')$ is a forest in Σ homoplastic to H.*

[$\phi(H)$ denotes the graph $(\phi(U(H)), \phi(V(H)))$.]

Proof. Since $U(J)$ is H-normal, $\phi^{-1}(H)$ is a forest in Σ'. Let H' be homoplastic to $\phi^{-1}(H)$ in Σ'. Suppose that $\phi(H')$ is not a forest. Since H' is a forest, there is a sequence $P'_1, ..., P'_k$ of mutually vertex-disjoint paths of H', where $k \geqslant 1$, P'_i has distinct ends $s_i, t_i \in \phi^{-1}(U(J))$ $(1 \leqslant i \leqslant k)$, $\phi(t_i) = \phi(s_{i+1})$ $(1 \leqslant i < k)$, and $\phi(t_k) = \phi(s_1)$. Choose such a sequence with $k \geqslant 1$ minimum. Let T'_i be the component of H' containing P'_i $(1 \leqslant i \leqslant k)$. From the minimality of k, $T'_i \neq T'_j$ for $1 \leqslant i < j \leqslant k$. Since H' is homoplastic to $\phi^{-1}(H)$, for $1 \leqslant i \leqslant k$ there is a path P_i of $\phi^{-1}(H)$ from s_i to t_i, and $P_1, ..., P_k$ all belong to different components of $\phi^{-1}(H)$. Hence $\phi(P_1) \cup \cdots \cup \phi(P_k)$ includes a circuit of H, a contradiction. Thus $\phi(H')$ is a forest.

Let $\alpha: \Sigma' \to \Sigma'$ be a homeomorphism such that $\alpha(x) = x$ for all $x \in bd(\Sigma')$ and $\alpha(H')$ is homotopic to $\phi^{-1}(H)$. For $x \in \Sigma$, we define $\beta(x)$ as follows: we choose $y \in \phi^{-1}(x)$, and let $\beta(x) = \phi(\alpha(y))$. This is uniquely defined; for if $x \in bd(\Sigma) \cup U(J)$ then $\phi(\alpha(y)) = x$ since $\alpha(y) = y$, and if

$x \in \Sigma - (bd(\Sigma) \cup U(J))$ then y is unique. It is easy to verify that $\beta: \Sigma \to \Sigma$ is a homeomorphism, and $\beta(x) = x$ $(x \in bd(\Sigma))$, and $\beta(\phi(H'))$ is homotopic to H, as required.

Let $\Sigma \cong \Sigma(a, b, c)$ where $2a + 2b + c > 1$, and let $k \geqslant 0$ be an integer. Let w be the maximum of

$$w(\Sigma', 2k, 3(2a + b + c - 1))$$

(as in (3.6)), taken over all surfaces $\Sigma' \cong \Sigma(a, b, c')$, where $c' \leqslant 2(2a + b + c - 1)$. Let

$$\mu = 2(2a + b + c)(k + w) + 2k$$

$$v = 6(2a + b + c - 1)(w + 2\mu + 1) + 2k.$$

We define $v(\Sigma, k) = v$. One form of our main result is the following, and the remainder of this section is devoted to its proof. If C is a cuff of a surface Σ, we denote by $\Sigma + \hat{C}$ the surface obtained by pasting a closed disc onto C.

(4.6) *Let Σ be a connected surface with at least two cuffs and let $k \geqslant 0$ be an integer. Let G be a proper graph in Σ, with $|V(G) \cap bd(\Sigma)| \leqslant 2k$, such that*

(i) *for every G-normal I-arc B with $|V(G) \cap B| \leqslant v(\Sigma, k)$ the ends of B are in the same cuff of Σ,*

(ii) *for every G-normal O-arc B with $|V(G) \cap B| \leqslant v(\Sigma, k)$ either B is null-homotopic, or for some cuff C, B is null-homotopic in $\Sigma + \hat{C}$ and $V(G) \cap C \subseteq B$, and*

(iii) *for every cuff C, $|V(G) \cap C| \geqslant 2$.*

Let H be a forest in Σ with $V(G) \cap bd(\Sigma) = V(H) \cap bd(\Sigma)$. Then H is G-feasible.

Proof. Let (Δ, Π) be a minimal net for Σ with respect to G; let J be the associated seam graph, and let $\phi: \Delta \to \Sigma$ be the associated surjection. Choose a, b, c such that $\Sigma \cong \Sigma(a, b, c)$ and let w, μ, v be as in the definition of $v(\Sigma, k)$.

Let $z \in V(H) \cap bd(\Sigma)$. Then $z \in C$ for some cuff C. We claim that there are $v - 2k$ paths of G, each between z and $bd(\Sigma) - C$, pairwise vertex-disjoint except for their ends. For suppose not, and let G' be the graph obtained from G by deleting every vertex of G in C except z. By (4.2) applied to G', there is a G'-normal O-arc F' with $F' \cap bd(\Sigma) = \emptyset$ and $|V(G') \cap F'| \leqslant v - 2k - 1$, such that F' is not null-homotopic. Choose F' with $|V(G') \cap F'|$ minimum. Then it is easy to see that $F' \cap r$ is connected, for every region r of G', and there is a G-normal O-arc F, homotopic to F', with

$$V(G) \cap F \subseteq (V(G') \cap F') \cup ((V(G) \cap C) - \{z\}).$$

Then $|V(G) \cap F| \leqslant v$, and so by (ii), either F is null-homotopic in Σ, or F is null-homotopic in $\Sigma + \hat{C}'$ for some cuff C' with $V(G) \cap C' \subseteq F$. Now F is not null-homotopic in Σ, since F' is not. On the other hand, $V(G) \cap C \nsubseteq F$ since $z \notin F$, and if C' is a cuff with $C' \neq C$ then $V(G) \cap C' \nsubseteq F$, because $V(G) \cap C' \neq \varnothing$ by (iii) and

$$V(G) \cap F \cap C' \subseteq V(G') \cap F' \cap C' = \varnothing.$$

This is a contradiction. We deduce that there are $v - 2k$ paths of G, each between z and $bd(\Sigma) - C$, pairwise vertex-disjoint except for their ends, as claimed.

Let $e_1, ..., e_{v-2k}$ be the edges of these paths incident with z. Then $\phi^{-1}(e_1), ..., \phi^{-1}(e_{v-2k})$ are edges of $\phi^{-1}(G)$, each incident with exactly one member of $\phi^{-1}(z)$. Now if $z \in V(J)$ then $|\phi^{-1}(z)| = d(z) - 1$ where $d(z)$ is the valency of z in J, counting loops twice. But

$$d(z) \leqslant 2\,|E(J)| \leqslant 6(2a+b+c-1)$$

by (2.4). Thus if $z \in V(J)$ then $|\phi^{-1}(z)| \leqslant 6(2a+b+c-1)$. But this inequality also holds if $z \notin V(J)$, since then $|\phi^{-1}(z)| = 1 \leqslant 6(2a+b+c-1)$. Hence at least

$$\frac{v-2k}{6(2a+b+c-1)} \geqslant \mu$$

of $\phi^{-1}(e_1), ..., \phi^{-1}(e_{v-2k})$ are incident in $\phi^{-1}(G)$ with the same member of $\phi^{-1}(z)$. Let us denote such a member of $\phi^{-1}(z)$ by $l(z)$.

For each edge e of J, we say that e is *long* if $|\bar{e} \cap V(G)| \geqslant 2\mu + w + 2$ and *short* otherwise. Every long edge of J is an edge of $J - bd(\Sigma)$, because

$$|V(G) \cap bd(\Sigma)| \leqslant 2k < 2\mu + w + 2.$$

Let e be a long edge of J, and let the vertices of G in $\bar{e}$ be $v_1, ..., v_r$, in order on $\bar{e}$. Then $r \geqslant 2\mu + w + 2$ since e is long. Let $W(e) = \{v_{\mu+1}, v_{r-\mu}\}$ and let $W = \bigcup(W(e))$, the union being taken over all long edges e of J. Let J^+ be the graph $(U(J), V(J) \cup W)$. Evidently each short edge of J is an edge of J^+, which we call a *short* edge of J^+. Every long edge of J is divided into three edges of J^+; the middle one we call a *central* edge, while the other two are *linking* edges. For each linking edge e of J^+, $|\bar{e} \cap V(G)| = \mu + 1$ while for each central edge e of J^+, $|e \cap V(G)| \geqslant w$. Let K be the subgraph of J^+ with vertex set W and with edges the central edges of J^+; and let L be the subgraph of J^+ obtained from J^+ by deleting the central edges and those edges which meet (and hence are subsets of) $bd(\Sigma)$.

Now Δ is obtained from Σ by cutting along $U(J-bd(\Sigma))=U(K)\cup U(L)$. We can do so in two stages, first cutting along $U(L)$ and then cutting along $U(K)$. Let Σ' be the surface obtained from Σ by cutting along $U(L)$. There are natural surjections $\theta:\Sigma'\to\Sigma$ and $\psi:\Delta\to\Sigma'$ where

$$\theta(\psi(x))=\phi(x) \qquad (x\in\Delta).$$

Then $\theta^{-1}(K)$ is a matching in Σ', and Δ is obtained from Σ' by cutting along $U(\theta^{-1}(K))$.

Now for each edge e of L, $|\bar{e}\cap V(G)|\leqslant w+2\mu+1$ because e is either short or linking. Thus

$$|V(G)\cap U(L)|\leqslant 6(2a+b+c-1)(w+2\mu+1)=v-2k$$

by (2.4). We claim that every O-arc $F\subseteq U(L)\cup bd(\Sigma)$ is a cuff of Σ. For F is G-normal, and

$$|V(G)\cap F|\leqslant|V(G)\cap U(L)|+|V(G)\cap bd(\Sigma)|\leqslant v$$

and so by (ii), there is a closed disc $R\subseteq\Sigma+\hat{C}$ with $bd(R)=F$, for some cuff C of Σ. But $F\subseteq U(J^+)$ and J^+ has only one region (as a graph in Σ), and so $R\cap\Sigma\subseteq bd(R)$, that is, $F=C$, as claimed.

It follows that $U(L)$ includes no O-arc. Suppose that $U(L)$ includes an I-arc. Then we may choose an I-arc $F\subseteq U(L)$ such that no point of F is in $bd(\Sigma)$ except its ends. Now

$$|V(G)\cap F|\leqslant|V(G)\cap U(L)|\leqslant v$$

and so by (i), the ends of F are in the same cuff C of Σ. Hence there is an O-arc $F'\subseteq F\cup C$ with $F\subseteq F'$; but then F' is not a cuff, and $F'\subseteq U(L)\cup bd(\Sigma)$, contrary to the result of the previous paragraph. We deduce that $U(L)$ includes no I-arc.

Since $U(L)$ includes no O-arc or I-arc of Σ, it follows that $\Sigma'\cong\Sigma(a,b,c')$ for some $c'\geqslant c$. The cuffs of Σ' are in 1–1 correspondence with the components of $U(L)\cup bd(\Sigma)$, and each of these components contains a vertex of J. Hence the number of cuffs of Σ' is at most $2(2a+b+c-1)$ by (2.4). Thus $c'\leqslant 2(2a+b+c-1)$, and from the definition of w we have

$$w\geqslant w(\Sigma',2k,3(2a+b+c-1)).$$

Moreover, $\theta^{-1}(K)$ is a matching in Σ'; and it has at most $3(2a+b+c-1)$ edges, by (2.4). Let

$$M_1 = \{\psi(l(z)) : z \in V(G) \cap bd(\Sigma)\}$$
$$M_2 = \theta^{-1}(V(G) \cap bd(\Sigma)) - M_1$$
$$M_3 = V(\theta^{-1}(G)) \cap bd(\Sigma') - (M_1 \cup M_2).$$

Hence M_1, M_2, M_3 are mutually disjoint and have union $V(\theta^{-1}(G)) \cap bd(\Sigma')$.

By (4.4), there is a forest H_1 in Σ, homotopic to H, such that $U(H_1) \cap U(L) \subseteq bd(\Sigma)$ and each member of M_2 is an isolated vertex of $\theta^{-1}(H_1)$. Let H_2 be the forest in Σ' obtained from $\theta^{-1}(H_1)$ by deleting M_2. Then

$$|V(H_2) \cap bd(\Sigma')| = |V(H_1) \cap bd(\Sigma)| \leqslant 2k$$

since $U(H_1) \cap U(L) \subseteq bd(\Sigma)$. Thus by (3.6) there is a forest H_3 in Σ', homoplastic to H_2, such that $|U(H_3) \cap U(\theta^{-1}(K))| \leqslant w$. Since each edge of $\theta^{-1}(K)$ passes through at least w vertices of $\theta^{-1}(G)$, we may choose H_3 so that in addition $U(H_3) \cap U(\theta^{-1}(K)) \subseteq V(\theta^{-1}(G))$. Let $H_4 = (U(H_3) \cup M_2, V(H_3) \cup M_2)$; then H_4 is homoplastic to $\theta^{-1}(H_1)$, and $|U(H_4) \cap U(\theta^{-1}(K))| \leqslant w$, and each member of M_2 is an isolated vertex of H_4. Let $H_5 = \theta(H_4)$; then by (4.5) H_5 is a forest in Σ homoplastic to H_1 and hence to H, and $U(H_5) \cap U(L) \subseteq bd(\Sigma)$, and $U(H_5) \cap U(K) \subseteq V(G)$, and every member of M_2 is an isolated vertex of $\theta^{-1}(H_5)$. Let $H_6 = (U(H_5) \cup V(G), V(H_5) \cup V(G))$. Since H_6 is homoplastic in Σ to H, we may replace H by H_6, for if the result holds for H_6 then it holds for H. In summary, then, we may assume that

(1) *$U(H) \cap U(J) \subseteq V(G) \subseteq V(H)$, there are at most w non-isolated vertices of H in $U(K)$, and for each $z' \in V(\theta^{-1}(H)) \cap bd(\Sigma')$, if z' is not an isolated vertex of $\theta^{-1}(H)$ then $z' = \psi(l(z))$ for some $z \in V(G) \cap bd(\Sigma)$.*

Now $\phi^{-1}(H)$ is a forest in Δ, and by (1), $V(\phi^{-1}(H)) \cap bd(\Delta) = V(\phi^{-1}(G) \cap bd(\Delta))$. If H' is a forest in Δ homoplastic in Δ to $\phi^{-1}(H)$, then $\phi(H')$ is a forest in Σ homoplastic in Σ to H by (4.5). Thus, to show that H is G-feasible it suffices to show that $\phi^{-1}(H)$ is $\phi^{-1}(G)$-feasible. To do so we use (4.1).

Let F be a $\phi^{-1}(G)$-normal proper I-arc of Δ with ends s, t, and let $F^* = F - \{s, t\}$. Let C_1, C_2 be the components of $bd(\Delta) - \{s, t\}$, and let Λ be the set of vertex sets of all components of $\phi^{-1}(H)$ which intersect both C_1 and C_2 and contain neither s nor t. By (4.1), it suffices to show that $|V(\phi^{-1}(G)) \cap F^*| \geqslant |\Lambda|$, that is, $|V(G) \cap \phi(F^*)| \geqslant |\Lambda|$.

We claim that $|\Lambda| \leqslant w+k$. For let M be the set of non-isolated vertices of $\theta^{-1}(H)$. Let $\lambda \in \Lambda$, and $d \in \lambda$. Then since $|\lambda| \geqslant 2$, d is not an isolated vertex of $\phi^{-1}(H)$, and hence $d \in \psi^{-1}(v)$ for some $v \in M$. Moreover, $d \in bd(\Delta)$ and so $v \in U(\theta^{-1}(K)) \cup bd(\Sigma')$. If $v \in U(\theta^{-1}(K))$ then $|\psi^{-1}(v)| = 2$, and if $v \in bd(\Sigma') - U(\theta^{-1}(K))$ then $|\psi^{-1}(v)| = 1$. It follows that

$$\begin{aligned}\left|\bigcup (\lambda : \lambda \in \Lambda)\right| &\leqslant \left|\bigcup (\psi^{-1}(v) : v \in M \cap (U(\theta^{-1}(K)) \cup bd(\Sigma')))\right| \\ &\leqslant 2\,|M \cap U(\theta^{-1}(K))| + |M \cap bd(\Sigma')| \\ &\leqslant 2w + 2k\end{aligned}$$

by (1). Since $|\lambda| \geqslant 2$ for each $\lambda \in \Lambda$, we deduce that $|\Lambda| \leqslant w+k$ as claimed. Hence we may assume that

$$|V(G) \cap \phi(F^*)| \leqslant k + w - 1.$$

By (2.6) there is a G-normal O-arc A in Σ with $\phi(F) \subseteq A \subseteq \phi(F) \cup U(J)$ and with

$$|V(G) \cap A| \leqslant 2(2a+b+c)(k+w) + 2k = \mu \leqslant v.$$

By hypothesis (ii) either A is null-homotopic in Σ, or for some cuff C, A is null-homotopic in $S + \hat{C}$ and $V(G) \cap C \subseteq A$. Suppose A is not null-homotopic in Σ, so that the second alternative applies. Since F is proper and $C \subseteq bd(\Sigma) \subseteq \phi(bd(\Delta))$, it follows that $\phi(F^*) \cap C = \emptyset$ and so $\phi(F^*)$ is contained in a single component B of $A - C$. Then $\bar{B}$ is an I-arc, since by (iii) $|V(G) \cap C| \geqslant 2$, and $\phi(F) \subseteq \bar{B}$. Let the ends of $\bar{B}$ be s', t'. Since A is null-homotopic in $\Sigma + \hat{C}$ but not in Σ, it follows that $C - \{s', t'\}$ has two components C_1', C_2' such that $A \cap C \subseteq \bar{C}_1'$ and $C_2' \cup \bar{B}$ is a null-homotopic O-arc in Σ. Since $V(G) \cap C \subseteq A$ and $A \cap C \subseteq \bar{C}_1'$, it follows that $V(G) \cap C_2' = \emptyset$. Hence $C_2' \cup \bar{B}$ satisfies our original hypothesis for A. Thus we may choose A to be null-homotopic in Σ.

Now one component of $\Sigma - A$ is homeomorphic to an open disc since A is null-homotopic; and the other is not, since $\Sigma \not\cong \Sigma(0,0,0)$. Let S be the first component. Let $J' = (U(J), V(J) \cup \{\phi(s), \phi(t)\})$. Since F is proper and $\phi(bd(\Delta)) = U(J)$, it follows that $\phi(F^*) \cap U(J) = \emptyset$ and consequently, since $A \subseteq \phi(F) \cup U(J)$, we infer that $A \cap U(J) = U(Q)$ for some path Q in J' between $\phi(s)$ and $\phi(t)$. It follows that for every edge e of J', either $e \subseteq S$ or $e \cap S = \emptyset$. Suppose that $e \subseteq S$ for some edge e of J'. There is a path P of J' with distinct ends such that every edge of P is included in S. Choose P with as many edges as possible, and let its ends be u, v. Now J' has only one region in Σ, and so not both $u, v \in A$; we assume that $u \in S$. For the same reason, there is only one edge of J' incident with u and with some vertex in $V(P) - \{u\}$. But by (2.3), there is another edge of J' incident with u, and it is not a loop, since J' has only one region. This contradicts the maximality of P. We deduce that no edge of J' intersects S, and hence no vertex of J' lies in S. Thus $U(J) \cap S = \varnothing$.

Let R_1, R_2 be the two components of $\Delta - F$, where $R_i \cap bd(\Delta) = C_i$ $(i = 1, 2)$. Then $\phi(R_1) - A$, $\phi(R_2) - A$ are the two components of $\Sigma - A$, and we assume that $\phi(R_1) - A = S$. Since $U(J) \cap S = \varnothing$, it follows that ϕ maps R_1 and its boundary $C_1 \cup F$ homeomorphically onto S, A, respectively. We deduce that

(2) $\phi(C_1) = A - \phi(F) = U(Q) - \{\phi(s), \phi(t)\}$.

We may assume that $\Lambda \neq \varnothing$. Choose $\lambda \in \Lambda$ and $d \in \lambda \cap C_1$. We claim that

(3) $\phi(d) \in U(K)$.

For as before $\psi(d)$ is a non-isolated vertex of $\theta^{-1}(H)$, and $\psi(d) \in \theta^{-1}(U(K)) \cup bd(\Sigma')$. We suppose that $\phi(d) \notin U(K)$; hence $\psi(d) \notin \theta^{-1}(U(K))$, and so $\psi(d) \in bd(\Sigma')$. By (1), $\psi(d) = \psi(l(z))$ for some $z \in V(G) \cap bd(\Sigma)$. Hence $d = l(z)$, since ψ acts injectively on $\phi^{-1}(bd(\Sigma))$. By definition of $l(z)$, there are paths $P_1, ..., P_\mu$ of G such that

(a) each has initial vertex z and terminal vertex in some cuff of Σ not containing z,

(b) the paths are pairwise vertex-disjoint except for their ends,

(c) for each i, if e_i is the first edge of P_i then $\phi^{-1}(e_i)$ is incident with $l(z)$.

Now from (c), $\phi^{-1}(e_i) \subseteq R_1$, and from (a), P_i meets A in a vertex different from z, for each i (because $\phi(R_1) - A = S$, which is an open disc, and therefore cannot contain the terminal vertex of P_i). Since

$$|V(G) \cap A - \{z\}| < \mu,$$

A meets some cuff not containing z, from (b). Thus there is an I-arc $A' \subseteq A$ with ends in different cuffs. But

$$|V(G) \cap A'| \leqslant |V(G) \cap A| \leqslant \mu \leqslant v$$

contrary to (i). This proves (3).

Let e be a long edge of J with $\phi(d) \in e$. Let the vertices of G in $\bar{e}$ be $v_1, ..., v_r$ say, in order on $\bar{e}$. Choose i such that $\phi(d) = v_i$; then $\mu + 1 < i < r - \mu$ since $\phi(d)$ is in an edge of K. Since $|V(G) \cap U(Q)| \leqslant |V(G) \cap A| \leqslant \mu$ it follows that neither all of $v_1, ..., v_{\mu+1}$ nor all of $v_{r-\mu}, ..., v_r$ belong to $U(Q)$, but (2) gives

$$v_i = \phi(d) \in U(Q) - \{\phi(s), \phi(t)\} \subseteq U(J)$$

and furthermore $e \in E(J)$. Therefore $U(Q) \subseteq e$ and so, by (2) again, $\phi(C_1) \subseteq e$.

Now the net (Δ, Π) is minimal, and so if we replace the portion $U(Q)$ of e between $\phi(s)$ and $\phi(t)$ by $\phi(F)$ we do not obtain a "better" seam graph; that is,

$$|V(G) \cap \phi(F^*)| \geqslant |V(G) \cap \phi(C_1)|.$$

But

$$|V(G) \cap \phi(C_1)| = |V(\phi^{-1}(G)) \cap C_1| \geqslant |A|$$

since ϕ acts injectively on C_1, and so $|V(G) \cap \phi(F^*)| \geqslant |A|$. This completes the proof.

5. The Main Result—Second Version

The preliminary form (4.6) of our main result needs refinement. The hypotheses that $c \geqslant 2$ and that $|V(G) \cap C| \geqslant 2$ for each cuff C are unnatural, and were introduced for technical reasons—we shall show later how to remove them. But the principal defect of (4.6) is hypothesis (ii), which is too strong. In this section we replace it with a weaker condition.

A subset $X \subseteq \Sigma$ is *planar* if $X \subseteq \Delta$ for some closed disc $\Delta \subseteq \Sigma$. Thus an O-arc is planar if and only if it is null-homotopic. Let us say that $X \subseteq \Sigma$ is *solid* if X is closed and locally arc-connected (see Section 11). Thus, for example, if G is a proper graph in Σ, the closure of the union of some of the regions and some of the edges of G is solid.

We shall assume the following fact (implied by (11.2) and (11.10)).

(5.1) *If Σ is connected and $X \subseteq \Sigma$ is solid and $X \neq \Sigma$, then X is planar if and only if every O-arc included in X is null-homotopic.*

We say that $X \subseteq \Sigma$ is *near-planar* if either X is planar, or X is planar in $\Sigma + \hat{C}$, for some cuff C. We say that X *surrounds* cuff C (in Σ) if X is planar in $\Sigma + \hat{C}$ but not in Σ. It follows easily from (5.1) that

(5.2) *If $X \subseteq \Sigma$ is solid and X is planar in $\Sigma + \hat{C}$, where C is a cuff, then X surrounds C in Σ if and only if some O-arc included in X surrounds C.*

Let G be a proper graph in Σ. We define $\alpha(G)$ to be the minimum of $|X \cap V(G)|$, taken over all solid connected G-normal sets $X \subseteq \Sigma$ which are not near-planar. (It is easy to see that the minimum exists provided that $\Sigma \not\cong \Sigma(0, 0, c)$ for $c = 0$, 1, or 2.)

If G is proper in Σ, C is a cuff of Σ, and $r \geqslant 0$ is an integer, we define

$\mathscr{A}_r(C)$ to be the union of all solid connected G-normal sets $X \subseteq \Sigma$ which surround C and for which $|V(G) \cap X| \leqslant r$. If there is no such X, $\mathscr{A}_r(C)$ is defined to be $\varnothing$.

(5.3) *If $\Sigma \not\cong \Sigma(0, 0, 2)$ and G is a proper graph in Σ, and C, C' are distinct cuffs and r, $r' \geqslant 0$ are integers with $r + r' < \alpha(G)$ then $\mathscr{A}_r(C) \cap \mathscr{A}_{r'}(C') = \varnothing$.*

Proof. Suppose that $x \in \mathscr{A}_r(C) \cap \mathscr{A}_{r'}(C')$. Choose $X \subseteq \Sigma$ with $x \in X$ such that X is solid, connected, and G-normal and such that X surrounds C and $|V(G) \cap X| \leqslant r$; and choose X' similarly for C', r'. It is easy to see that, because $\Sigma \not\cong \Sigma(0, 0, 2)$, $X \cup X'$ is not near-planar; but $X \cup X'$ is solid (by (11.6)), connected, and G-normal, and

$$|V(G) \cap (X \cup X')| \leqslant r + r' < \alpha(G),$$

a contradiction. Thus there is no such x, as required.

If X is a topological space and $X' \subseteq \Sigma$ is homeomorphic to X, we say that X' is an *X-arc*. We define the *ends* of a $[0, 1]$-arc in the natural way. We need the following topological fact.

(5.4) *Let F, F_0 be O-arcs in Σ, where F_0 is null-homotopic. Let $r > 0$, and let $v_1, ..., v_r \in F$ be distinct, and in order on F. Let the components of $F - \{v_1, ..., v_r\}$ be $A_1, ..., A_r$, where $\{v_{i-1}, v_i\} \subseteq \bar{A}_i$ $(1 \leqslant i \leqslant r)$. (Throughout, subscripts should be read modulo r.) Let $P_1, ..., P_r$ be $[0, 1]$-arcs, such that for $1 \leqslant i \leqslant r$, v_i is one end of P_i and the other end (u_i say) is in F_0. For $1 \leqslant i \leqslant r$, let $B_i = \varnothing$ if $u_{i-1} = u_i$, and be a component of $F_0 - \{u_{i-1}, u_i\}$ if $u_{i-1} \neq u_i$. Suppose that $P_{i-1} \cup P_i \cup A_i \cup B_i$ is planar $(1 \leqslant i \leqslant r)$. Then F is null-homotopic.*

We also need the following variation on (5.1).

(5.5) *Let G be a proper graph in a connected surface Σ, and let Z be the closure of the union of some of the regions of G, with $Z \neq \Sigma$. Then Z is planar if and only if every G-normal O-arc included in Z is null-homotopic.*

We postpone the proof until Section 11.

Let G be a proper graph in Σ. If $x, y \in \Sigma$ are distinct, we define $d(x, y)$ to be $\min |U(G) \cap X|$, the minimum being taken over all $[0, 1]$-arcs X with ends x, y. It is easy to see that $d(x, y)$ exists if Σ is connected; and that if x, $y \notin U(G) - V(G)$ the minimum is attained by a $[0, 1]$-arc X which is G-normal. If $x = y$, we define $d(x, y) = 0$ if $x \notin U(G)$, and $d(x, y) = 1$ if $x \in U(G)$. If

$x \in \Sigma$ and $A \subseteq \Sigma$ is non-empty, we define $d(x, A)$ to be min $d(x, y)$, taken over all $y \in A$, and for any integer $t \geqslant 0$ we define

$$Y_t(A) = \{x \in \Sigma : d(x, A) \leqslant t\}.$$

(5.6) *Let Σ be a connected surface such that $\Sigma \not\cong \Sigma(0, 0, c)(c = 0, 1, 2)$, and let G be a proper graph in Σ. Let C be a cuff of Σ, and let F_0 be a G-normal O-arc surrounding C. Let t be an integer with*

$$|V(G) \cap F_0| \leqslant t \leqslant \tfrac{2}{9}\alpha(G) - 1.$$

Then $Y_t(F_0)$ is planar in $\Sigma + \hat{C}$.

Proof. Let Z be the closure of $Y_t(F_0)$. Then Z is clearly the closure of the union of some of the regions of G, and so it suffices, by (5.5), to show that every G-normal O-arc $F \subseteq Z$ is null-homotopic in $\Sigma + \hat{C}$. Let F be such an O-arc, and let the vertices of G on F be $v_1, ..., v_r$, in order on F. For $1 \leqslant i \leqslant r$, let P_i be a G-normal $[0, 1]$-arc with one end v_i and the other end, u_i say, in F_0, and with $|V(G) \cap P_i| \leqslant t + 1$. (This exists since $d(v_i, F_0) \leqslant t + 1$.) Let $A_1, ..., A_r$ and $B_1, ..., B_r$ be as in (5.4); where if $u_{i-1} \neq u_i$, choose B_i to be a component of $F_0 - \{u_{i-1}, u_i\}$ with

$$|V(G) \cap B_i| \leqslant \tfrac{1}{2} |V(G) \cap F_0|.$$

Now for $1 \leqslant i \leqslant r$, $C_i = A_i \cup B_i \cup P_{i-1} \cup P_i$ is solid (by (11.6)), connected, and G-normal; and

$$|V(G) \cap C_i| \leqslant 0 + \tfrac{1}{2} |V(G) \cap F_0| + (t+1) + (t+1) \leqslant \tfrac{5}{2}t + 2 < \alpha(G)$$

and so C_i is near-planar. Suppose that C' is a cuff surrounded by C_i. Then $C_i \subseteq \mathscr{A}_{(5/2)t+2}(C')$, and $C_i \subseteq \mathscr{A}_{2t+1}(C)$, as is easily seen. But $(\tfrac{5}{2}t + 2) + (2t + 1) < \alpha(G)$ and so $C' = C$, by (5.3). Thus C_i is near-planar, and surrounds no cuff except possibly C; and so C_i is planar in $\Sigma + \hat{C}$. By (5.4) we deduce that F is null-homotopic in $\Sigma + \hat{C}$, as required.

If $\Sigma \cong \Sigma(a, b, c)$ where $3a + 3b + c \geqslant 3$, and F is a null-homotopic O-arc, then there is a unique closed disc $\Delta \subseteq \Sigma$ with boundary F. If $x \in \Delta - F$ we say that x is *inside* F (in Σ). If F is an O-arc surrounding a cuff C and $x \in \Sigma$, we say that x is *inside* F in Σ if it is inside F in $\Sigma + \hat{C}$. If $X \subseteq \Sigma$, we say that X is *inside* F if every $x \in X$ is inside F.

(5.7) *Let $\Sigma \cong \Sigma(a, b, c)$, where $3a + 3b + c \geqslant 3$, and let G be a proper graph in Σ. Let C be a cuff, and let $r \geqslant 0$ be an integer with $r \leqslant \tfrac{2}{9}\alpha(G) - 1$. Then $\mathscr{A}_r(C)$ is planar in $\Sigma + \hat{C}$.*

Proof. Choose a G-normal O-arc F surrounding C with $|V(G) \cap F| \leqslant r$, such that as many vertices and edges of G as possible are inside F. (This is possible by (5.2) unless $\mathscr{A}_r(C) = \emptyset$ in which case the theorem is trivial.) Let Δ be the subset of $\Sigma + \hat{C}$ homeomorphic to $\Sigma(0, 0, 1)$ with boundary F.

Now $\Delta \cup Y_r(F)$ is planar in $\Sigma + \hat{C}$, by (5.6), and so it suffices to prove that $\mathscr{A}_r(C) \subseteq \Delta \cup Y_r(F)$. Suppose then that $x \in \mathscr{A}_r(C) - \Delta$. Choose $X \subseteq \Sigma$ with $x \in X$ such that X is solid, connected, and G-normal, and X surrounds C, and $|V(G) \cap X| \leqslant r$. By (5.2), there exists an O-arc $F' \subseteq X$ which surrounds C, and F' is clearly G-normal, because X is. Suppose that $X \cap F = \emptyset$. Then F is inside F', because $x \notin \Delta$; and so from our choice of F, $V(G) \cap F = \emptyset$ and no vertex or edge of G lies in the portion of Σ between F and F'. But then it is easy to see (from (11.4)) that $d(x, F) = d(x, F') \leqslant r$ and so $x \in Y_r(F)$. We may assume then that $X \cap F \neq \emptyset$; but then clearly $d(x, F) \leqslant r$ (again, from (11.4)), and again $x \in Y_r(F)$. This completes the proof.

We shall need the following, which is essentially Theorem (4.1) of [2].

(5.8) *Let Σ be a cylinder and let C_1, C_2 be the two cuffs. Let G be a proper graph in Σ. Let $r, s \geqslant 0$ be integers. Suppose that*

(i) *every G-normal O-arc F with $|F \cap V(G)| < r$ is null-homotopic, and*

(ii) *every G-normal I-arc F with $|F \cap V(G)| < s$ has both its ends in the same cuff.*

Then there are mutually vertex-disjoint paths $P_1, ..., P_r$ of G, each between C_1 and C_2, and mutually vertex-disjoint circuits $B_1, ..., B_s$ of G, none null-homotopic, such that for $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant s$ the intersection of P_i and B_j is a path.

Let G be a proper graph in Σ. We say that G is *boundary-linked* if for every cuff C of Σ and every G-normal O-arc F surrounding C, $|V(G) \cap F| \geqslant |V(G) \cap C|$. We now turn to a revised form of our main result.

(5.9) *Let $\Sigma \cong \Sigma(a, b, c)$, where $c \geqslant 2$ and $(a, b, c) \neq (0, 0, 2)$. Let $k \geqslant 0$ be an integer. Let G be a boundary-linked proper graph in Σ with $|V(G) \cap bd(\Sigma)| \leqslant 2k$ and with $\alpha(G) > 9kv(\Sigma, k)$, such that for every cuff C, $|V(G) \cap C| \geqslant 2$. Let H be a forest in Σ with $V(G) \cap bd(\Sigma) = V(H) \cap bd(\Sigma)$. Then H is G-feasible.*

Proof. Let $v = v(\Sigma, k)$, and let C be a cuff. We define $k(C) = |V(G) \cap C|$. Then, since Σ has at least two cuffs, $2 + k(C) \leqslant 2k$, and so $2 \leqslant k(C) \leqslant 2k - 2$. In particular, $k \geqslant 2$. Now if $C' \neq C$ is a cuff, then $C' \subseteq \mathscr{A}_{2k}(C')$, and so by (5.3), $C' \cap \mathscr{A}_{2kv-1}(C) = \emptyset$, since

$2k+(2kv-1)<\alpha(G)$. Hence $\mathscr{A}_{2kv-1}(C)\cap bd(\Sigma)\subseteq C$. But $\mathscr{A}_{2kv-1}(C)$ is planar in $\Sigma+\hat{C}$, by (5.7), and so there exists a closed disc $\Delta(C)\subseteq\Sigma+\hat{C}$, with $\mathscr{A}_{2kv-1}(C)\subseteq\Delta(C)$ and $\Delta(C)\cap bd(\Sigma)=C$. Let $A(C)$ be a G-normal O-arc surrounding C with $|V(G)\cap A(C)|\leqslant vk(C)$, chosen in addition so that as many vertices of G as possible are inside $A(C)$. Let Z be the set of all points of Σ inside $A(C)$. Then $A(C)\subseteq\mathscr{A}_{2kv-1}(C)$, since $A(C)$ surrounds C, and so $A(C)\subseteq\Delta(C)$. In particular, $Z\subseteq\Delta(C)$.

Let $\beta=\frac{1}{2}v+k(C)+2$, and let j be an integer with $1\leqslant j\leqslant\beta$. Let W be $(\Sigma-U(G))\cup V(G)$, so that W is the union of $V(G)$ with all the regions of G. Let $y\in Y_j(A(C))\cap W$. Then either $y\in A(C)$, or there is a G-normal $[0,1]$-arc F with one end y and the other in $A(C)$, with $|V(G)\cap F|\leqslant j$. In either case, there is a solid, connected, G-normal set $X\subseteq\Sigma$ with $y\in X$ and $A(C)\subseteq X$, with

$$|V(G)\cap X|\leqslant j+|V(G)\cap A(C)|\leqslant\tfrac{1}{2}v+k(C)+2+vk(C)\leqslant 2kv-1<\alpha(G)$$

since $k(C)\leqslant 2k-2$. Hence X surrounds C, and so $X\subseteq\mathscr{A}_{2kv-1}(C)\subseteq\Delta(C)$. We deduce that

$$Y_j(A(C))\cap W\subseteq\Delta(C).$$

But $Y_j(A(C))$ is included in the closure of $Y_j(A(C))\cap W$, and $\Delta(C)$ is closed, and so $Y_j(A(C))\subseteq\Delta(C)$. We define Z_j to be the closure of $Y_j(A(C))\cup Z$. Then $Z_j\subseteq\Delta(C)$, since $\Delta(C)$ is closed, and it is easy to see that Z_j is the closure of the union of some of the regions of G, and $Z_j\cap bd(\Sigma)=C$. Moreover, Z_j has no "cut-vertices," and hence there is a circuit C_j of G with $U(C_j)\subseteq Z_j$, such that $Z_j-U(C_j)$ is inside the O-arc $U(C_j)$. We observe that $C_1, C_2, ..., C_\beta$ are mutually vertex-disjoint.

Now there is no G-normal $[0,1]$-arc F in Σ with one end in $U(C_1)$ and the other in $U(C_\beta)$ with $|V(G)\cap F|<\beta$, because such F must meet all of $C_1, C_2, ..., C_\beta$. Moreover, there is no G-normal O-arc F surrounding C with $U(C_1)-F$ inside F and $F-U(C_\beta)$ inside $U(C_\beta)$ and with

$$|V(G)\cap F|<vk(C),$$

because of our choice of $A(C)$. Thus, we may apply (5.8) to the portion of Σ between C_1 and C_β to deduce that there are mutually vertex-disjoint paths $P_1, ..., P_{vk(C)}$ of G, each with one end in $V(C_1)$, the other in $V(C_\beta)$, and no other vertex in $V(C_1)\cup V(C_\beta)$; and mutually vertex-disjoint circuits $B_1(C), ..., B_\beta(C)$ of G, with $B_1(C)=C_1$, $B_\beta(C)=C_\beta$, such that for $1\leqslant i<j\leqslant\beta$, $B_i(C)$ is inside $B_j(C)$; and the intersection of each P_i with each $B_j(C)$ is a path.

We define $N(C)=B_{k(C)+1}(C)$. Now each of $P_1, ..., P_{vk(C)}$ meets $N(C)$

in a path, and so we may choose mutually vertex-disjoint subpaths $N_1(C), ..., N_{k(C)}(C)$ of $N(C)$, such that

(i) $V(N_1(C)) \cup \cdots \cup V(N_{k(C)}(C)) = V(N(C))$, and

(ii) for $1 \leqslant i \leqslant k(C)$, exactly v of $P_1, ..., P_{vk(C)}$ intersect $N(C)$ in a subpath of $N_i(C)$.

Suppose that $I \subseteq \{1, ..., k(C)\}$ and $V' \subseteq V(G)$, with $|I| + |V'| < k(C)$. Put

$$X = V' \cup \bigcup (V(N_i(C)) : i \in I).$$

We claim that there is a path of G between C and $N(C)$ avoiding X. Now $|V'| < k(C)$, and so one of $B_1(C), ..., B_{k(C)}(C)$ does not meet X; $B_s(C)$ say. But since G is boundary-linked, there exist $k(C)$ mutually vertex-disjoint paths of G, each between C and $B_s(C)$ and disjoint from $N(C)$ (by a planar form of Menger's theorem, or by (5.8) with $s=0$ applied to the portion of Σ between C and $B_s(C)$). Since fewer than $k(C)$ elements of X are not in $N(C)$, it follows that there is a path of G between C and $B_s(C)$ which avoids X. Moreover, there exists $r \in \{1, ..., k(C)\} - I$, since $|I| < k(C)$. Then v of $P_1, ..., P_{vk(C)}$ meet $N(C)$ only in $N_r(C)$ and $|V'| < k(C) < v$; thus at least one of $P_1, ..., P_{vk(C)}$ avoids X. It follows that there is a path of G between C and $N(C)$ avoiding X, as claimed.

Because of the above, we may deduce from Menger's theorem, applied to the graph obtained from G by contracting $N_i(C)$ to a single vertex for each i with $1 \leqslant i \leqslant k(C)$, that there are $k(C)$ mutually vertex-disjoint paths $Q_1, ..., Q_{k(C)}$ of G, such that for each i, Q_i has one end in C, the other in $V(N_i(C))$, and all its other vertices and edges are inside $U(N(C))$. For each $x \in V(G) \cap C$, let $t(x)$ denote the vertex of $N(C)$ such that one of $Q_1, ..., Q_{k(C)}$ has ends x and $t(x)$.

Let $H(C)$ be a G-normal O-arc surrounding C, with $V(G) \cap H(C) = V(N(C))$, such that the points of $U(G)$ inside $H(C)$ are the points of $U(G)$ inside $U(N(C))$ together with the points in the edges of $N(C)$. Then $H(C) \subseteq Y_\beta(A(C)) \subseteq \mathscr{A}_{2kv-1}(C)$, and $4kv - 2 < \alpha(G)$ and so by (5.3) if C, C' are distinct cuffs, $H(C) \cap H(C') = \emptyset$ and neither is inside the other.

Let Σ' be the component which is not a cylinder of the surface obtained from Σ by cutting along $H(C)$ for every cuff C. Loosely, we shall regard Σ' as a subset of Σ. Now let us take the restriction of G to Σ', and for each cuff C of Σ and for all $x \in V(G) \cap C$, let us move each vertex of $N_i(C)$ along $bd(\Sigma')$ in the appropriate direction until it becomes identified with $t(x)$, where $t(x) \in V(N_i(C))$. (Edges incident with the vertices we are moving must also be moved in the natural way, to remain incident with the moving vertex.) Let the resulting graph be G'. Then G' is a proper graph in Σ'.

Let $\phi : \Sigma \to \Sigma'$ be a homeomorphism such that $\phi(C) = H(C)$ for each cuff

C, and $\phi(x)=t(x)$ $(x \in V(G) \cap C)$. Let $H'=\phi(H)$; then H' is a forest in Σ', and if it is G'-feasible then H is G-feasible. To show that H' is G'-feasible we shall show that conditions (i) and (ii) of (4.6) hold; and it is easy to see that they are respectively equivalent to the claims which follow.

Claim (a). *For any G-normal $[0, 1]$-arc F in Σ with $F \subseteq \Sigma'$, and with both ends in $bd(\Sigma')$, if $|V(G) \cap F| \leqslant v$ then both ends of F lie in $H(C)$ for some cuff C of Σ.*

Suppose that F satisfies these hypotheses, and yet its ends lie in $H(C)$, $H(C')$, respectively, where C, C' are different cuffs. Let A be the O-arc $U(B_\beta(C))$. Now $H(C')$ does not meet A; for $A \subseteq \mathscr{A}_{2kv-1}(C)$ and $H(C') \subseteq \mathscr{A}_{2kv-1}(C')$, and by (5.3) these two sets are disjoint. $H(C')$ is not inside A, because it surrounds C' and A does not. Thus no point of $H(C')$ is in or inside A. But $H(C)$ is inside A, and so F meets A in some vertex u say. Since $u \in Z_\beta$ and $u \notin \bar{Z}$, it follows that $u \in Y_\beta(A(C)) \subseteq Y_{\beta+1}(A(C))$, and so there is a G-normal $[0, 1]$-arc F_0 with one end u and the other in $A(C)$, and with

$$|V(G) \cap F_0| \leqslant \beta + 1 = \tfrac{1}{2}v + k(C) + 3.$$

Choose similarly a G-normal $[0, 1]$-arc F_0' with one end in F and the other in $A(C')$ and with

$$|V(G) \cap F_0'| \leqslant \tfrac{1}{2}v + k(C') + 3.$$

Put $X = A(C) \cup F_0 \cup F \cup F_0' \cup A(C')$. Then X is solid, connected, G-normal, not near-planar, and

$$\begin{aligned}|V(G) \cap X| &\leqslant vk(C) + (\tfrac{1}{2}v + k(C) + 3) + v + (\tfrac{1}{2}v + k(C') + 3) + vk(C') \\ &\leqslant 2kv + 2v + 2k + 6 \leqslant 9kv < \alpha(G),\end{aligned}$$

a contradiction. This proves claim (a).

Claim (b). *For any G-normal O-arc F in Σ with $F \subseteq \Sigma'$, and with*

$$|\{(C, i)\colon C \text{ is a cuff of } \Sigma,\ 1 \leqslant i \leqslant k(C), \text{ and } F \text{ meets } N_i(C)\}|$$
$$+\,|F \cap (V(G) - bd(\Sigma'))| \leqslant v$$

either F is null-homotopic in Σ, or F surrounds some cuff C of Σ and F meets $N_i(C)$ for all i $(1 \leqslant i \leqslant k(C))$.

Suppose F satisfies these hypotheses, and is not null-homotopic in Σ. Let L be the first term in the inequality above. We suppose first that $L=0$, so that $V(G) \cap F \cap bd(\Sigma') = \emptyset$ and $|V(G) \cap F| \leqslant v$. Since $\alpha(G) > 9kv \geqslant v$, F is near-planar in Σ. Since it is not null-homotopic, it surrounds some cuff C say. But $A(C)$ is inside $N(C)$ and hence inside F, contrary to our choice of $A(C)$. Hence $L \neq 0$, and so

(1) $|F \cap (V(G) - bd(\Sigma'))| < v$.

Choose a cuff C so that $F \cap V(N(C)) \neq \emptyset$. Now, with β as before, $v = 2(\beta - k(C) - 2)$ and so F does not have $\geqslant 2$ vertices in common with each of $B_{k(C)+2}(C), ..., B_{\beta-1}(C)$; hence F does not meet $B_\beta(C)$. But F surrounds C, and $B_1(C)$ is inside F, and so F meets every path of G between $B_1(C)$ and $B_\beta(C)$. Choose i with $1 \leqslant i \leqslant k(C)$. There are v mutually vertex-disjoint paths of G between $B_1(C)$ and $B_\beta(C)$, which meet $N(C)$ only in $N_i(C)$; and so F must meet $N_i(C)$, by (1). Since this holds for each value of i $(1 \leqslant i \leqslant k(C))$, claim (b) is true.

This completes the proof of the theorem.

6. Allowing Smaller Cuffs

In this section we remove the condition that $|V(G) \cap C| \geqslant 2$ for each cuff C, with the following version of our main result.

(6.1) *Let $\Sigma \cong \Sigma(a, b, c)$, where $c \geqslant 2$ and $(a, b, c) \neq (0, 0, 2)$. Let $k \geqslant 0$ be an integer. Let G be a boundary-linked proper graph in Σ with $|V(G) \cap bd(\Sigma)| \leqslant 2k - 2c$ and with $\alpha(G) > 9kv(\Sigma, k)$. Let H be a forest in Σ with $V(G) \cap bd(\Sigma) = V(H) \cap bd(\Sigma)$. Then H is G-feasible.*

Proof. Let g be the number of cuffs C of Σ with $|V(G) \cap C| \leqslant 1$. We may replace the hypothesis

$$|V(G) \cap bd(\Sigma)| \leqslant 2k - 2c$$

by the weaker hypothesis

$$|V(G) \cap bd(\Sigma)| \leqslant 2k - 2g$$

and the same conclusion will hold, as we now show. We proceed by induction on g. If $g = 0$ the result is true by (5.9). We assume then that $g > 0$. Let C be a cuff with $|V(G) \cap C| \leqslant 1$. Let us choose a G-normal O-arc A' surrounding C with $|V(G) \cap A'| \leqslant 1$, such that as many vertices and edges of G as possible are included in the union of A' with its inside. Then $A' \subseteq \mathscr{A}_1(C)$, and so by (5.3) $A' \cap \mathscr{A}_1(C') = \emptyset$ for every cuff $C' \neq C$. Therefore there is a vertex of G not in A' or its inside. The maximality condition in our choice of A' now implies that $|V(G) \cap A'| = 1$ and that if r is the region of G with $A' \subseteq \bar{r}$, there is a vertex of G incident with r, not in A' or its inside. It follows that we may choose a G-normal O-arc A surrounding C, with $A' - A$ inside A, and with $|V(G) \cap A| = 2$. Now A meets no cuff $C' \neq C$; for if it did, $A \cup C'$ would be solid, connected, G-normal, and not near-planar, and yet

$$|V(G) \cap (A \cup C')| \leqslant 2k - 2g + 2 \leqslant 9kv(\Sigma, k) < \alpha(G),$$

which is impossible. Thus we may cut Σ along A and obtain a new surface, with one component Σ' which is homeomorphic to Σ. Loosely, we regard Σ' as a subset of Σ. Let G' be the restriction of G to Σ'. Then G' is boundary-linked in Σ', by choice of A'. Let $V(G) \cap A = \{a_1, a_2\}$, where if $V(G) \cap C \neq \emptyset$ there is a path of G between C and a_1 not using a_2. (If $V(G) \cap C \neq \emptyset$ there is such a path, because G is boundary-linked.)

By replacing H by a homotopic forest if necessary, we may assume that $a_1, a_2 \in V(H)$, and $U(H) \cap A = \{a_1, a_2\}$, and that if $V(G) \cap C = \emptyset$ then $U(H) \subseteq \Sigma'$, and if $V(G) \cap C \neq \emptyset$ then there is a path of H between a_1 and C not using a_2. Let H' be the restriction of H to Σ'; then H' is a forest in Σ'. But $V(H') \cap bd(\Sigma') = V(G') \cap bd(\Sigma')$, and

$$|V(H') \cap bd(\Sigma')| \leqslant |V(H) \cap bd(\Sigma)| + 2 \leqslant 2k - 2g + 2 = 2k - 2g',$$

where g' is the number of cuffs of Σ' containing at most one vertex of G'. By our inductive hypothesis H' is G'-feasible, and so H is G-feasible as required.

7. Adding New Cuffs

In this section we eliminate the condition $c \geqslant 2$. The natural way to do so is to add another cuff, if our surface has only one, by making a small cut in the surface in as unobtrusive a way as possible. We need to know that such a cut can be made without reducing $\alpha(G)$ too much, and that is the main topic of this section.

Let $\Sigma \cong \Sigma(a, b, c)$, where $3a + 3b + c \geqslant 3$, and let G be a proper graph in Σ. Let C be a cuff of Σ. If r is an integer with $1 \leqslant r \leqslant \frac{2}{9}\alpha(G) - 1$, then by (5.7), $\mathscr{A}_r(C)$ is planar in $\Sigma + \hat{C}$. Moreover, if $\mathscr{A}_r(C) \neq \emptyset$ there is an O-arc $A_r(C) \subseteq bd(\Sigma) \cup U(G)$ which surrounds C, such that $A_r(C) \subseteq \mathscr{A}_{r+1}(C)$ and $\mathscr{A}_r(C)$ is inside $A_r(C)$ (for the closure of $\mathscr{A}_r(C)$ has no "cut-vertices" of the relevant type, as is easily seen). Let $\Delta_r(C)$ be the set of all points of Σ inside $A_r(C)$. We define Δ_r to be the union of $\Delta_r(C)$ taken over all cuffs C with $\mathscr{A}_r(C) \neq \emptyset$.

If e is an edge of G, let $G \backslash e$ be the graph obtained from G by deleting e. Let Σ_e be the surface obtained from Σ by cutting along $\bar{e}$. Let ϕ_e be the associated surjection. Then $\phi_e^{-1}(G \backslash e)$ is a proper graph in Σ_e; and provided $\bar{e} \cap bd\,\Sigma = \emptyset$ and e is not a loop, we have $\Sigma_e \cong \Sigma(a, b, c+1)$.

(7.1) *Suppose that Σ, G are as above, and $2 \leqslant r \leqslant \frac{2}{9}\alpha(G)$, and that e is an edge of G, not a loop, with $e \nsubseteq \Delta_{r-1}$ and with $\bar{e} \cap bd(\Sigma) = \emptyset$. Then $\alpha(\phi_e^{-1}(G \backslash e)) \geqslant r$.*

Proof. Let $G' = \phi_e^{-1}(G \backslash e)$. Suppose that $X \subseteq \Sigma_e$ is solid, connected, G'-

normal, and not near-planar in Σ_e. We must show that $|V(G')\cap X|\geqslant r$. We may assume that $X\cap bd(\Sigma_e)\subseteq V(G')$ as is easily seen. But then ϕ_e is 1–1 on X, and $\phi_e(X)$ is solid, connected, and G-normal in Σ. If $\phi_e(X)$ is not near-planar in Σ, then

$$|X\cap V(G')|=|\phi_e(X)\cap V(G)|\geqslant\alpha(G)\geqslant\tfrac{9}{2}r\geqslant r,$$

as required. We assume then that $\phi_e(X)$ is near-planar in Σ. If it is planar in Σ then it is near-planar in Σ_e, a contradiction; thus it surrounds some cuff C of Σ. Hence $\phi_e(X)\subseteq\Delta_{r'}(C)$, where $r'=|V(G)\cap\phi_e(X)|=|V(G')\cap X|$. Now X is not planar in $\Sigma_e+\hat{C}$, and so $e\subseteq\Delta_{r'}(C)$. But $e\nsubseteq\Delta_{r-1}$, and so $r'\geqslant r$, and $|V(G')\cap X|\geqslant r$, as required.

(7.2) *Let $\Sigma\cong\Sigma(a,b,c)$, where $3a+3b+c\geqslant 3$, and let G be a boundary-linked proper graph in Σ. Let e be an edge of G which is not a loop, such that $\bar{e}\cap bd(\Sigma)=\emptyset$, and there is no null-homotopic G-normal O-arc A with $|V(G)\cap A|\leqslant 1$ and with e inside A. Then $\phi_e^{-1}(G\backslash e)$ is boundary-linked in Σ_e.*

Proof. Put $G'=\phi_e^{-1}(G\backslash e)$. Let F be a G'-normal O-arc of Σ_e which surrounds some cuff C of Σ_e. We must show that $|V(G')\cap F|\geqslant|V(G')\cap C|$. We may clearly assume that $F\cap bd(\Sigma')\subseteq V(G')$. If $C=\phi_e^{-1}(\bar{e})$ then $\phi_e(F)$ is a G-normal null-homotopic O-arc of Σ, and e is inside it, and so by hypothesis, $|V(G)\cap\phi_e(F)|\geqslant 2$. But

$$|V(G')\cap F|=|V(G)\cap\phi_e(F)|$$

and $|V(G')\cap C|=2$ and the result is true. We assume then that $\phi_e(C)$ is a cuff of Σ. But then $\phi_e(F)$ is a G-normal O-arc of Σ surrounding $\phi_e(C)$, and so

$$|V(G)\cap\phi_e(F)|\geqslant|V(G)\cap\phi_e(C)|$$

and again the result is true.

(7.3) *Let $\Sigma\cong\Sigma(a,b,c)$ where $3a+3b+c\geqslant 3$. Let G be a proper graph in Σ, and let r be an integer with $|V(G)\cap C|\leqslant r-1$ for every cuff C of Σ, and with $2\leqslant r\leqslant\frac{2}{9}\alpha(G)$. Then there is an edge e of G which is not a loop, such that $e\nsubseteq\Delta_{r-1}$ and $\bar{e}\cap bd(\Sigma)=\emptyset$, and such that there is no G-normal null-homotopic O-arc F with $|V(G)\cap F|\leqslant 1$ and with e inside F.*

Proof. We proceed by induction on $|E(G)|$. If there is a G-normal null-homotopic O-arc F with $|V(G)\cap F|\leqslant 1$ and with some edge inside it, let G' be the graph obtained from G by deleting all vertices and edges inside F. It is easy to see that $\alpha(G')=\alpha(G)$, and that if the result is true for G' then it is true for G. But it is true for G' by our inductive hypothesis.

We assume then that there is no such O-arc F. From this and the fact that $\alpha(G) \geqslant 9$ it follows that for every loop e of G, $\bar{e}$ surrounds some cuff. Since $\alpha(G) \neq 0$ it follows that G has an edge which is not a loop, e_1 say. If $e_1 \nsubseteq \Delta_{r-1}$ then e_1 satisfies the theorem, and we assume that $e_1 \subseteq \Delta_{r-1}$. Let C be a cuff with $\mathscr{A}_{r-1}(C) \neq \emptyset$ and $e_1 \subseteq \Delta_{r-1}(C)$. Let $A_{r-1}(C)$ be defined as at the start of this section. If $A_{r-1}(C) \subseteq bd(\Sigma)$ then $A_{r-1}(C)$ is a cuff $C' \neq C$, since $e_1 \subseteq \Delta_{r-1}(C)$, and so $\Sigma \cong \Sigma(0, 0, 2)$, a contradiction. Thus $A_{r-1}(C) \nsubseteq bd(\Sigma)$, and hence there is an edge e of G with $e \subseteq A_{r-1}(C)$. We claim that $e \nsubseteq \Delta_{r-1}$. For certainly $e \nsubseteq \Delta_{r-1}(C)$; we suppose $e \subseteq \Delta_{r-1}(C')$ for some cuff $C' \neq C$. Then $A_{r-1}(C) \cap \Delta_{r-1}(C') \neq \emptyset$, but $A_{r-1}(C) \nsubseteq \Delta_{r-1}(C')$, since $A_{r-1}(C)$ is not null-homotopic in $\Sigma + \hat{C}'$. Hence $A_{r-1}(C) \cap A_{r-1}(C') \neq \emptyset$. But $A_{r-1}(C) \subseteq \mathscr{A}_r(C)$, and $A_{r-1}(C') \subseteq \mathscr{A}_r(C')$, and so $\mathscr{A}_r(C) \cap \mathscr{A}_r(C') \neq \emptyset$, contrary to (5.3). This proves our claim that $e \nsubseteq \Delta_{r-1}$. We suppose e is a loop, so that $A_{r-1}(C) = \bar{e}$. But then both regions incident with e are subsets of $\mathscr{A}_1(C)$, and so $\mathscr{A}_1(C)$ is not inside $A_{r-1}(C)$, a contradiction, since $r \geqslant 2$. Thus e is not a loop. Finally, we claim that $\bar{e} \cap bd(\Sigma) = \emptyset$. For suppose $v \in V(G)$ is an end of e, and $v \in C'$ for some cuff C'. Since $|V(G) \cap C'| \leqslant r-1$ it follows that $\mathscr{A}_{r-1}(C')$ includes every region of G incident with v, and so $e \subseteq \Delta_{r-1}(C') \subseteq \Delta_{r-1}$, a contradiction. Thus $\bar{e} \cap bd(\Sigma) = \emptyset$, and so e satisfies the theorem.

Putting these results together, we obtain

(7.4) *Let $\Sigma \cong \Sigma(a, b, c)$, where $3a + 3b + c \geqslant 3$, and let G be a boundary-linked proper graph in Σ with $\alpha(G) \geqslant \frac{9}{2}r$, where $r \geqslant 2$ is an integer, and $|V(G) \cap C| \leqslant r-1$ for every cuff C. Then there is an edge e of G which is not a loop, with $\bar{e} \cap bd(\Sigma) = \emptyset$, such that $\phi_e^{-1}(G \backslash e)$ is boundary-linked in Σ_e and*

$$\alpha(\phi_e^{-1}(G \backslash e)) \geqslant r.$$

As a first application of (7.4), we deduce the following final form of our main result. (A second application appears in Section 9.)

(7.5) *Let $\Sigma \cong \Sigma(a, b, c)$, where $3a + 3b + c \geqslant 3$. Let $k \geqslant 0$ be an integer, and let $v = v(\Sigma(a, b, c+1), k)$. Let G be a boundary-linked proper graph in Σ with $\alpha(G) \geqslant 42kv$ and with $|V(G) \cap bd(\Sigma)| \leqslant 2k - 2c - 2$. Let H be a forest in Σ with $V(G) \cap bd(\Sigma) = V(H) \cap bd(\Sigma)$. Then H is G-feasible.*

Proof. If $c \geqslant 2$ this follows from (6.1), since $v(\Sigma, k) \leqslant v$. If $V(H) \cap bd(\Sigma) = \emptyset$ the theorem is trivial. We assume then that $V(H) \cap bd(\Sigma) \neq \emptyset$ and $c \leqslant 1$. It follows that $c = 1$ and $k \geqslant 3$. Let $r = \lfloor 28kv/3 \rfloor$. Then $\alpha(G) \geqslant 9r/2$, and we may apply (7.4). Let e be an edge of G satisfying the conclusions of (7.4) with ends a, b. Then $\alpha(\phi_e^{-1}(G\backslash e)) \geqslant r > 9kv$. We may assume that $U(H) \cap \bar{e} = \emptyset$. Let $a = \phi_e(a')$, $b = \phi_e(b')$, and define

$$H' = (U(\phi_e^{-1}(H)) \cup \{a', b'\},\ V(\phi_e^{-1}(H)) \cup \{a', b'\}).$$

By (6.1), H' is $\phi_e^{-1}(G\backslash e)$-feasible, and so H is G-feasible, as required.

8. Schisms

We wish to develop a more concrete definition of $\alpha(G)$, for use in applications. Let X_1, X_2, X_3 be the topological spaces of the graphs G_1, G_2, G_3 defined as follows. G_1 has exactly two vertices (u, v say) and three edges, one a loop on u, one a loop on v, and the other joining u, v; G_2 has exactly one vertex and two edges; and G_3 has two vertices and three edges, mutually parallel.

We say that $X \subseteq \Sigma$ is *schismatic* if X is either an O-arc or an X_i-arc for some i, and X is not near-planar, and every proper subset of X which is an O-arc is near-planar. It is easy to see that if X is schismatic and $X \cong X_1$ or X_2 then the two O-arcs included in X surround distinct cuffs; while if $X \cong X_3$, then $\Sigma \cong \Sigma(0, 0, 3)$ and all three O-arcs surround distinct cuffs.

It is easy to verify, using (5.1) and (5.2), that if $X \subseteq \Sigma$ is solid and connected, then X is near-planar if and only if no subset of X is schismatic. Thus $\alpha(G)$ equals the minimum of $|V(G) \cap X|$, taken over all G-normal schismatic sets X; and this provides us with an alternative definition of $\alpha(G)$.

In applications of our main theorem we might attempt to deal with cases where $\alpha(G)$ is too small by cutting the surface along the offending schismatic set. However, this does not work nicely; for instance, doing so may fail to produce a surface because the number of components produced may be infinite, and there are other inelegancies as well. It is more convenient to cut along a "schism," a minimal set cutting along which simplifies the surface.

Let X_4 be the topological space of the graph G_4 consisting of two vertices and two edges, with exactly one loop. The *end* of an X_4-arc is defined to be the point representing the monovalent vertex of G_4. By a *schism* in a surface $\Sigma \cong \Sigma(a, b, c)$ where $3a + 3b + c \geqslant 3$ we mean a subset of Σ which is one of the following:

(i) An O-arc F which is not near-planar, with $|F \cap bd(\Sigma)| \leqslant 1$,

(ii) a proper I-arc with its ends in distinct cuffs,

(iii) a proper I-arc F with its ends in the same cuff C, such that $F \cup C$ is not near-planar,

(iv) an X_1-arc or X_2-arc such that both its O-arcs surround distinct cuffs of Σ, and which contains no point of $bd(\Sigma)$,

(v) an X_3-arc such that all three of its O-arcs surround distinct cuffs, and which contains no point of $bd(\Sigma)$,

(vi) an X_4-arc with its end in one cuff and its O-arc surrounding another cuff, containing only one point of $bd(\Sigma)$.

If G is a proper graph in Σ, let us define $\omega(G)$ to be the minimum of $|V(G) \cap X|$ taken over all G-normal schisms X in Σ. Then

$$\alpha(G) - |V(G) \cap bd(\Sigma)| \leqslant \omega(G) \leqslant \alpha(G)$$

because if X is a schism then $X \cup bd(\Sigma)$ includes a schismatic set, and any schismatic set includes a schism. Thus, for example, we can replace the condition $\alpha(G) \geqslant 42kv$ in (7.5) by the condition $\omega(G) \geqslant 42kv$. This form of (7.5) is often more convenient for use in applications.

9. Minors

We turn now to applications. The first is to the theory of graph minors. A *digraph* (G, G^+, G^-) in a surface Σ consists of a graph G in Σ and two functions $G^+, G^-: E(G) \to V(G)$ such that for every edge e of G, $\{G^+(e), G^-(e)\} = \bar{e} - e$. The digraph is *proper* if G is proper. Let (G, G^+, G^-), (H, H^+, H^-) be proper digraphs in Σ. We say that (H, H^+, H^-) is a *boundary-rooted minor* of (G, G^+, G^-) if $V(H) \cap bd(\Sigma) = V(G) \cap bd(\Sigma)$ and if for each vertex v of H there is a non-null connected subgraph $\phi(v)$ of G, and for each edge e of H there is an edge $\phi(e)$ of G, such that

(i) for distinct $v, v' \in V(H)$, $\phi(v)$ and $\phi(v')$ are vertex-disjoint,

(ii) for distinct $e, e' \in E(H)$, $\phi(e)$ and $\phi(e')$ are distinct,

(iii) for $v \in V(H)$ and $e \in E(H)$, $\phi(e)$ is not an edge of $\phi(v)$,

(iv) for $e \in E(H)$, $G^+(\phi(e))$ is a vertex of $\phi(H^+(e))$, and $G^-(\phi(e))$ is a vertex of $\phi(H^-(e))$, and

(v) for $v \in V(H) \cap bd(\Sigma)$, v is a vertex of $\phi(v)$.

(9.1) *Let $\Sigma \cong \Sigma(a, b, c)$ where $3a + 3b + c \geqslant 3$. Let (H, H^+, H^-) be a proper digraph in Σ. Let g be the number of isolated vertices of H which are not in $bd(\Sigma)$. Let k be an integer with*

$$2k \geqslant 4\,|E(H)| + 4g + |V(H) \cap bd(\Sigma)| + 2c + 2$$

and let c_0 be an integer with $c_0 \geqslant c + |E(H)| + g + 1$. Let (G, G^+, G^-) be a proper digraph in Σ such that

(i) $V(G) \cap bd(\Sigma) = V(H) \cap bd(\Sigma)$,

(ii) *G is boundary-linked, and*

(iii) $\alpha(G) \geqslant 42k5^{(g+|E(H)|)}\nu(\Sigma(a, b, c_0), k)$.

Then (H, H^+, H^-) *is a boundary-rooted minor of* (G, G^+, G^-).

Proof. If $g \neq 0$, let us add a (directed) loop to an isolated vertex of H not in $bd(\Sigma)$. If the theorem is true for this new digraph it is true for (H, H^+, H^-); and the new digraph satisfies all the hypotheses of the theorem (leaving k and c unchanged). Thus, by repeating this procedure we may eliminate all such isolated vertices of H.

We assume then that $g = 0$. Let $E(H) = \{f_1, ..., f_r\}$. By $r = |E(H)|$ applications of (7.4), we can find distinct edges $e_1, ..., e_r$ of G which are not loops such that, for $i = 1, ..., r$,

(i) e_i has no end in $bd(\Sigma_{i-1})$,

(ii) $\theta_i^{-1}(G_i)$ is boundary-linked in Σ_i, and

(iii) $\alpha(\theta_i^{-1}(G_i)) \geqslant 42k5^{r-i}\nu(\Sigma(a, b, c_0), k)$,

where Σ_i is the surface obtained from Σ by cutting along $\bar{e}_1, ..., \bar{e}_i$ and θ_i is the associated surjection and G_i is the graph obtained from G by deleting $e_1, ..., e_i$ $(i = 0, 1, ..., r)$. It follows from (i) that no two of $e_1, ..., e_r$ have a common end.

Let $\psi: \Sigma \to \Sigma$ be a homeomorphism of Σ which fixes every point of $bd(\Sigma)$ and maps the middle third of f_i onto the whole of e_i $(1 \leqslant i \leqslant r)$. By replacing (H, H^+, H^-) by its image under ψ, we may assume that $\bar{e}_i \subseteq f_i$ and the directions of e_i and f_i coincide $(1 \leqslant i \leqslant r)$. Let $H' = (U(H) - Y, V(H) \cup Z)$ where $Y = e_1 \cup \cdots \cup e_r$ and Z is the set of all ends of $e_1, ..., e_r$. Then H' is a forest in Σ, and its components are in a natural correspondence with $V(H)$. Hence $\theta_r^{-1}(H')$ is a forest in Σ_r. Moreover,

$$V(\theta_r^{-1}(G_r)) \cap bd(\Sigma_r) = V(\theta_r^{-1}(H')) \cap bd(\Sigma_r),$$

and

$$\begin{aligned} |V(\theta_r^{-1}(H')) \cap bd(\Sigma_r)| &= 2\,|E(H)| + |V(H) \cap bd(\Sigma)| \\ &\leqslant 2k - 2\,|E(H)| - 2c - 2 = 2k - 2c' - 2, \end{aligned}$$

where c' is the number of cuffs of Σ_r. By (7.5), $\theta_r^{-1}(H')$ is $\theta_r^{-1}(G_r)$-feasible. But that implies the conclusion of the theorem, as required.

If $bd(\Sigma) = \emptyset$ and (H, H^+, H^-), (G, G^+, G^-) are digraphs in Σ, and the first is a boundary-rooted minor of the second, we say that it is a *minor* of the second. We have immediately from (9.1) the following, a form of which was stated without proof in [3].

(9.2) *Let $\Sigma \cong \Sigma(a, b, 0)$ where $a+b \geqslant 1$, and let (H, H^+, H^-) be a digraph in Σ. Then there is a number N such that (H, H^+, H^-) is a minor of every digraph (G, G^+, G^-) in Σ with $\alpha(G) \geqslant N$.*

In a future paper we shall need a form of (9.1) which applies when Σ is a sphere, disc, or cylinder, and we now develop such a form. If G is a proper graph in a surface Σ, and $X, Y \subseteq \Sigma$ are disjoint, we define $k(X, Y)$ to be the maximum value of k such that there are paths $P_1, ..., P_k$ of G, mutually disjoint, and each with initial vertex in X and terminal vertex in Y. We begin with the cylinder case.

(9.3) *Let Σ be a cylinder with cuffs C_1 and C_2. Let (H, H^+, H^-) be a proper digraph in Σ. Let $t = |V(H)| + |E(H)|$, and let $r = 84(t+2)\, 5^t v \times (\Sigma(0, 0, t+4), 2t+4)$. Let (G, G^+, G^-) be a proper digraph in Σ such that $V(G) \cap bd(\Sigma) = V(H) \cap bd(\Sigma)$. Suppose that there are disjoint G-normal O-arcs F_1, F_2 with the following properties:*

(i) *F_1, F_2 are not null-homotopic, and C_1, F_2, F_1, C_2 occur on Σ in that order (in the natural sense),*

(ii) *$|V(G) \cap F| \geqslant 2r$ for every G-normal $[0, 1]$-arc F with $F \cap F_1 \neq \emptyset \neq F \cap F_2$,*

(iii) *$k(F_1, F_2) \geqslant r$, and*

(iv) *$k(F_i, C_i) \geqslant |V(G) \cap C_i|$ $(i = 1, 2)$.*

Then (H, H^+, H^-) is a boundary-rooted minor of (G, G^+, G^-).

Proof. Choose a G-normal $[0, 1]$-arc I with $I \cap F_1 \neq \emptyset \neq I \cap F_2$ and with $|V(G) \cap I|$ minimum. Choose $z \in I - U(G)$ such that the two components of $I - \{z\}$ both contain at least r vertices. Let I_1, I_2 be the two components of $I - \{z\}$, numbered so that I_j meets F_{3-j} $(j = 1, 2)$. It follows from the choice of I that $|V(G) \cap R| \geqslant r$ for every G-normal $[0, 1]$-arc R which for some $j (j = 1$ or $2)$ meets both F_j and I_j. For any two disjoint non-null-homotopic O-arcs $A_1, A_2 \subseteq \Sigma$, let $\Sigma(A_1, A_2)$ denote the closure of the portion of Σ between A_1 and A_2. Let Δ be a closed disc with $z \in \Delta \subseteq \Sigma$ such that $\Delta \cap (U(G) \cup bd(\Sigma)) = \emptyset$. Let C_3 be $bd(\Delta)$, and let Σ' be the surface obtained from Σ by removing the interior of Δ. Then $\Sigma' \cong \Sigma(0, 0, 3)$, and (G, G^+, G^-) is a proper digraph in Σ'. Moreover, by replacing (H, H^+, H^-) by an isomorphic copy (under an isomorphism which fixes all elements of $V(H) \cap bd(\Sigma)$) we may arrange that $U(H) \cap \Delta = \emptyset$, so that (H, H^+, H^-) is also a proper digraph in Σ'.

Claim 1. *G is boundary-linked in Σ'.*

For suppose that $F \subseteq \Sigma'$ is a G-normal O-arc surrounding some cuff C_i, where $i = 1, 2$, or 3, and that

$$|V(G) \cap F| < |V(G) \cap C_i|.$$

Now $V(G) \cap C_3 = \emptyset$, and so $i \neq 3$, and without loss of generality we may asssume $i = 1$. We have

$$k(F_1, C_1) \geqslant |V(G) \cap C_1|$$

by hypothesis, and so $F \nsubseteq \Sigma(F_1, C_1)$. But $F \nsubseteq \Sigma(F_1, C_2)$ since F surrounds C_1, and $\Delta \subseteq \Sigma(F_1, F_2)$; thus $F \cap F_1 \neq \emptyset$. Moreover, $F \cap (F_2 \cup I_1) \neq \emptyset$, since F surrounds C_1; and so there is a G-normal $[0, 1]$-arc $F' \subseteq F$ with

$$F' \cap F_1 \neq \emptyset \neq F' \cap (F_2 \cup I_1).$$

Now

$$\begin{aligned} |V(G) \cap F'| \leqslant |V(G) \cap F| < |V(G) \cap C_1| \leqslant |V(G) \cap bd(\Sigma)| \\ = |V(H) \cap bd(\Sigma)| \leqslant t \leqslant r. \end{aligned}$$

But if $F' \cap F_2 \neq \emptyset$ then $|V(G) \cap F'| \geqslant r$ by hypothesis; and if $F' \cap I_1 \neq \emptyset$ then $|V(G) \cap F'| \geqslant r$ by our earlier observation. In either case we obtain a contradiction. It follows that such F does not exist, and so claim 1 is true.

Claim 2. *As a drawing in Σ', G satisfies $\alpha(G) \geqslant r$.*

For let $X \subseteq \Sigma'$ be G-normal and schismatic. Since $\Sigma' \cong \Sigma(0, 0, 3)$, X is not an O-arc. It follows that there are two O-arcs $A_1, A_2 \subseteq X$ surrounding distinct cuffs of Σ'. We assume without loss of generality that A_1 surrounds C_1. Now A_2 surrounds either C_2 or C_3, and in either case

$$A_2 \cap (\Sigma(F_1, C_2) \cup I_2) \neq \emptyset.$$

But $A_1 \nsubseteq \Sigma(F_1, C_2)$ since A_1 surrounds C_1, and so $X \cap (F_1 \cup I_2) \neq \emptyset$. We must show that $|V(G) \cap X| \geqslant r$. If $X \cap F_2 \neq \emptyset$ then X includes an I-arc which intersects both F_2 and $F_1 \cup I_2$, and so $|V(G) \cap X| \geqslant r$ as claimed. If $X \cap F_2 = \emptyset$ then $X \subseteq \Sigma(F_2, C_2)$. But A_1 surrounds C_1 and so $A_1 \cap I_1 \neq \emptyset$. If $A_1 \cap F_1 \neq \emptyset$ then A_1 includes an I-arc which intersects both F_1 and I_2, and so $|V(G) \cap X| \geqslant r$ by our earlier observation. Finally, if $A_1 \cap F_1 = \emptyset$ then $A_1 \subseteq \Sigma(F_1, F_2)$ and so $|V(G) \cap A_1| \geqslant r$ by hypothesis (iii). Thus in every case $|V(G) \cap X| \geqslant r$. This proves Claim 2.

Because of Claims 1 and 2, we can apply (9.1) with Σ, a, b, c, c_0, k replaced by Σ', $0, 0, 3, t+4, 2t+4$, respectively; and we deduce that (H, H^+, H^-) is a boundary-rooted minor of (G, G^+, G^-) in Σ', and therefore also in Σ, as required.

There remain the disc and the sphere. They are both easily dealt with using (9.3).

(9.4) *Let Σ be a closed disc with cuff C and let (H, H^+, H^-) be a proper digraph in Σ. Let r be defined as in* (9.3). *Let (G, G^+, G^-) be a proper digraph in Σ, such that $V(G) \cap bd(\Sigma) = V(H) \cap bd(\Sigma)$. Suppose that there are disjoint G-normal O-arcs F_1, F_2 of Σ, with F_1 inside F_2, such that*

(i) $|V(G) \cap F| \geqslant 2r$ *for every G-normal* $[0, 1]$*-arc F with* $F \cap F_1 \neq \emptyset \neq F \cap F_2$,

(ii) $k(F_1, F_2) \geqslant r$, *and*

(iii) $k(F_1, C) \geqslant |V(G) \cap C|$.

Then (H, H^+, H^-) is a boundary-rooted minor of (G, G^+, G^-).

Proof. Choose a closed disc $\Delta \subseteq \Sigma$ with Δ inside F_1 and $\Delta \cap U(G) = \emptyset$. The result follows by applying (9.3) to the surface obtained from Σ by removing the interior of Δ, and to an isomorphic copy of (H, H^+, H^-) in this surface.

(9.5) *Let Σ be a sphere, and let (H, H^+, H^-) be a proper digraph in Σ. Let r be defined as in* (9.3). *Let (G, G^+, G^-) be a proper digraph in Σ. Suppose that there are disjoint G-normal O-arcs F_1, F_2 of Σ satisfying* (i) *and* (ii) *of* (9.4). *Then (H, H^+, H^-) is a boundary-rooted minor of (G, G^+, G^-).*

Proof. Choose a closed disc $\Delta \subseteq \Sigma$ with $\Delta \cap F_2 = \emptyset$ and $\Delta \cap U(G) = \emptyset$, such that Δ and F_1 lie in different components of $\Sigma - F_2$. The result follows by applying (9.4) to the surface obtained from Σ by removing the interior of Δ, and to an isomorphic copy of (H, H^+, H^-) in this surface.

As a consequence of (9.5) we have the following, an undirected form of which was stated without proof in [3].

(9.6) *Let (G, G^+, G^-), (H, H^+, H^-) be directed graphs in a sphere Σ, where G is isomorphic to an $N \times N$-grid. If N is sufficiently large* (*bounded below by a function of H*) *then (H, H^+, H^-) is a minor of (G, G^+, G^-).*

[The $N \times N$*-grid* is the adjacency graph of the squares of a chessboard with N^2 squares.]

10. An Algorithm

In this section we describe how our main result (7.5) can be used to give an algorithm to test if a given forest H is G-feasible, where G is a graph in a surface Σ. For fixed Σ and fixed H, the running time of the algorithm is bounded by a polynomial in the size of G. (However, it is not a practical algorithm—its existence is of interest mainly from the point of view of the theory of NP-completeness.)

The essential idea is that we test if our Theorem (7.5) can be applied. If so, the forest is G-feasible. If not, then either

(i) Σ is disconnected; we can consider its components separately, or

(ii) $\Sigma \cong \Sigma(0, 0, c)$ where $c \leqslant 2$; for these cases an algorithm was given in [3], or

(iii) G is not boundary-linked; in which case we can cut Σ along the relevant O-arc which is too short, and reduce to several problems on a surface homeomorphic to Σ, but with $|V(G) \cap bd(\Sigma)|$ reduced by at least one, or

(iv) $\omega(G)$ as defined in Section 8 is too small; in which case we can cut Σ along the offending schism, and reduce to several problems on a surface simpler than Σ.

In cases (iii) and (iv), the number of problems to which we reduce is a function of Σ and $|V(H) \cap bd(\Sigma)|$ alone, and does not depend on the size of G; and each of these problems is solvable in polynomial time in the size of G by (say) induction on $3a+2b+c$, and for fixed $3a+2b+c$ by induction on $|V(H) \cap bd(\Sigma)|$ (where $\Sigma \cong \Sigma(a, b, c)$). We need to check

(a) that before the above procedure we may arrange Σ, H, G so that G is proper in Σ and

$$V(G) \cap bd(\Sigma) = V(H) \cap bd(\Sigma),$$

and that this property is preserved under the reductions;

(b) that in case (iv), when we cut along a schism, the quantity "$3a+2b+c$" is smaller, for each component of the new surface, than it was for Σ,

(c) that in case (iii), the reduction does indeed decrease the quantity $|V(H) \cap bd(\Sigma)|$ by at least one, for each component of the new surface, and

(d) that in cases (iii) and (iv), the solution of the original problem is equivalent to the solution of a bounded number of problems on the new surface.

Let us turn then to the reductions used in (iii) and (iv). We treat them together. Let $X \subseteq \Sigma$ be either a schism or an O-arc surrounding some cuff, and let X be G-normal; and in addition let

$$X \cap bd(\Sigma) \subseteq V(G).$$

(In cases (iii) and (iv) we can always choose X to have this additional property.) Let Σ' be the surface obtained by cutting along X. Let $\phi: \Sigma' \to \Sigma$ be the associated surjection, and let $G' = \phi^{-1}(G)$ be defined as usual.

Let H' be a forest in Σ' with $V(H') \cap bd(\Sigma') = V(G') \cap bd(\Sigma')$. Then $\phi(H')$ may or may not be a forest in Σ; but up to homoplasty in Σ' there are only finitely many (bounded by a function of Σ and H) forests H' such that $\phi(H')$ is a forest in Σ homoplastic to H. (This is easily seen, via (3.4).) Moreover, H is G-feasible if and only if some such H' is G'-feasible. (Again, this is easily seen.) Thus our original problem is reduced to solving "is $H'G'$-feasible?" for all these finitely many possibilities for H'.

From these comments (a)–(d) may be verified. Finally, we must check that we can decide which of cases (i)–(iv) hold in polynomial time; but that is straightforward, and we omit the details.

A more natural problem than the one we have just solved is: let H be a forest in a surface Σ. Given a graph G in Σ, is there a subgraph of G which is a forest homotopic to H? In [4] we solved this when Σ is a sphere, disc, or cylinder, but we do not know how to solve it in general—indeed, solving it is open[1] even where H has just one edge and $\Sigma \cong \Sigma(0, 0, c)$ for general c.

Since we have solved the homoplasty problem, it follows that, as we claimed in the introduction, for any surface Σ and any integer k, there is a polynomially bounded algorithm which, given as input a graph G in Σ and vertices $s_1, t_1, ..., s_k, t_k$ of G, decides if there exist k vertex-disjoint paths of G, linking s_i and t_i $(1 \leqslant i \leqslant k)$, respectively. For we may cut small new cuffs in Σ, each meeting $U(G)$ only in one vertex of G, so that $s_1, t_1, ..., s_k, t_k$ each lie on one of the cuffs; we may extend the surface slightly, so that G is proper, and no vertex of G except $s_1, t_1, ..., s_k, t_k$ lies in the boundary of the surface; and then we test for the G-feasibility of all the (finitely many, up to homoplasty) matchings in which s_i is adjacent to t_i $(1 \leqslant i \leqslant k)$.

In [4, 6] a discussion is given of the complexity of the disjoint paths problem in general. In [6] we shall implicitly use (7.5) to give an algorithm to decide, given a graph $G = (V, E)$ (*not* required to be embedded in a surface) and vertices $s_1, t_1, ..., s_k, t_k \in V(G)$, whether there are k vertex-disjoint paths linking s_i and t_i $(1 \leqslant i \leqslant k)$. This has running time $O(|V|^2 \cdot |E|)$ for fixed k, which is much more efficient than the algorithm we have just described.

11. Appendix

We have assumed a number of results from point-set topology, and for

[1] *Note added in proof.* This problem has recently been solved by A. Schrijver, in "Disjoint circuits of prescribed homotopies in a graph on a surface," manuscript (1987).

most of them we have not been able to locate suitable references. We therefore sketch their proofs here. If $\phi: A \to B$ is a function and $X \subseteq A$, $\phi(X)$ denotes $\{\phi(x): x \in X\}$. We abbreviate $\phi(A)$ by $|\phi|$. We require the following.

(11.1) *If Σ is a surface and $\phi: [0, 1] \to \Sigma$ is continuous with $\phi(0) \neq \phi(1)$, there is a $[0, 1]$-arc included in $|\phi|$ with ends $\phi(0), \phi(1)$.*

For a proof, see, for example, [7, (4.2.5)].

If Σ is a surface and $Z \subseteq \Sigma$, we say that Z is *flat* if every continuous $\phi: S^1 \to Z$ is null-homotopic in Σ. We apply (11.1) to deduce

(11.2) *If Σ is a surface, and $\phi: S^1 \to \Sigma$ is continuous and non-null-homotopic, there is a non-null-homotopic O-arc included in $|\phi|$.*

Proof. Let $\Delta_i (i \in I)$ be an open cover of Σ such that for $i, i' \in I$, $\Delta_i \cup \Delta_{i'}$ is flat. If $\psi: S^1 \to \Sigma$ is continuous and non-null-homotopic, and $|\psi| \subseteq |\phi|$, we may choose, by compactness, a finite subset $K_\psi \subseteq S^1$ such that for every component X of $S^1 - K_\psi$ there exists $i \in I$ with $\psi(\bar{X}) \subseteq \Delta_i$. Let us choose ψ and K_ψ with $|K_\psi|$ minimum. Let χ be the set of components of $S^1 - K_\psi$. We see that

(1) $|K_\psi| \geqslant 3$; for by hypothesis, the union of any two Δ_i's is flat, and

(2) if $X, X' \in \chi$ and $\bar{X} \cap \bar{X}' = \emptyset$ then $\psi(\bar{X}) \cap \psi(\bar{X}') = \emptyset$.

Moreover, by (11.1) we may choose ψ such that the restriction to $\bar{X}$ is an injection, for each $X \in \chi$, since each Δ_i is flat. Take an orientation Ω of S^1, and let $X \in \chi$. Let X' be the next member of χ (under Ω). Let $x \in \bar{X}$ be the first member of X (under Ω) such that $\psi(x) = \psi(x')$ for some $x' \in \bar{X}'$. Now the restriction of ψ to the portion of $\overline{X \cup X'}$ between x and x' is null-homotopic, by our choice of the Δ_i's, and so this portion of ψ may be "removed"; that is, we may choose ψ such that $x = x'$. If we repeat this process for all $X \in \chi$ we find that the resultant $|\psi|$ is an O-arc, as required.

A subset $Z \subseteq \Sigma$ is *connected* if there do not exist non-empty $A_1, A_2 \subseteq Z$ with $A_1 \cup A_2 = Z$ and with $\bar{A}_1 \cap \bar{A}_2 \cap Z = \emptyset$. We say $Z \subseteq \Sigma$ is *arc-connected* if for every pair $a, b \in Z$ with $a \neq b$ there is a $[0, 1]$-arc included in Z with ends a, b.

(11.3) *Let Σ be a connected surface, let $\Delta \subseteq \Sigma$ be a closed disc, and let $Y \subseteq \Sigma$ be such that $Y \cap bd(\Delta)$ is arc-connected. Then any continuous map $\phi: S^1 \to Y \cup (\Delta - bd(\Delta))$ is homotopic to a map $\psi: S^1 \to Y$ with $|\psi| \subseteq |\phi| \cup bd(\Delta)$. In particular, if Y is flat then so is $Y \cup (\Delta - bd(\Delta))$.*

Proof. The second statement follows from the first. To prove the first, we may assume that Σ is a topological subspace of $\mathbb{R}^n$, and that

$$\Delta = \{(x, y, 0, ..., 0) : x^2 + y^2 \leqslant 1\}.$$

For distinct $x, y \in bd(\Delta)$, let $F(x, y) \subseteq bd(\Delta)$ be an I-arc with ends x, y which is included in a semicircle. We define $F(x, x) = \{x\}$ for $x \in bd(\Delta)$. We observe

(1) *For $x, y \in bd(\Delta)$ and $u, v \in F(x, y)$, $\|u - v\| \leqslant \|x - y\|$.*

($\| \ \|$ denotes Euclidean distance.)

(2) *There exists $\varepsilon_0 > 0$ such that for all $x, x' \in S^1$, if $\phi(x), \phi(x') \in bd(\Delta)$ and $\|\phi(x) - \phi(x')\| < \varepsilon_0$ then $F(\phi(x), \phi(x')) \subseteq Y$.*

For if $bd(\Delta) \subseteq Y$ this is trivial. If not, then since $|\phi|$ is closed, there is an open interval $I \subseteq bd(\Delta)$ with $I \nsubseteq Y$ and $I \cap |\phi| = \emptyset$. Let u, v be the ends of I; we may assume that $\bar{I} = F(u, v)$. Let $\varepsilon_0 = \|u - v\|$. Now suppose that $x, x' \in S^1$, with $\phi(x), \phi(x') \in bd(\Delta)$ and $\|\phi(x) - \phi(x')\| < \varepsilon_0$. We must show that $F(\phi(x), \phi(x')) \subseteq Y$. We may assume that $\phi(x) \neq \phi(x')$ since otherwise the result holds. Let F_1, F_2 be the two I-arcs in $bd(\Delta)$ with ends $\phi(x), \phi(x')$. Since $Y \cap bd(\Delta)$ is arc-connected we may assume that $F_1 \subseteq Y \cap bd(\Delta)$. Thus $I \nsubseteq F_1$, and so $I \subseteq F_2$, since $\phi(x), \phi(x') \notin I$. Since $\|u - v\| > \|\phi(x) - \phi(x')\|$ it follows from (1) that $F(\phi(x), \phi(x'))$ does not contain both u and v, and so $F_2 \neq F(\phi(x), \phi(z'))$. Hence $F_1 = F(\phi(x), \phi(x'))$ and the claim follows since $F_1 \subseteq Y$.

Let $Z = \{z \in S^1 : \phi(z) \in \Delta - bd(\Delta)\}$. We may assume that $Z \neq S^1$ since otherwise the theorem holds, and so each component of Z is an open interval, since Z is open. Let Ξ be the set of all these intervals. For $X \in \Xi$ with ends z, z', we may choose (by (2)) a continuous function $\psi_X : \bar{X} \to Y \cap bd(\Delta)$ such that

(i) $\psi_X(z) = \phi(z)$, $\psi_X(z') = \phi(z')$, and

(ii) if $\|\phi(z) - \phi(z')\| < \varepsilon_0$ then $|\psi_X| \subseteq F(\phi(z), \phi(z'))$.

For $X \in \Xi$, we define diam $(X) = \max(\|x - x'\| : x, x' \in \bar{X})$.

(3) *For all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $X \in \Xi$, if* $\mathrm{diam}(X) \leqslant \delta$ *then $\|\phi(x) - \psi_X(x)\| \leqslant \varepsilon$ for all $x \in X$.*

For given $\varepsilon > 0$, since ϕ is uniformly continuous there exists $\delta > 0$ such that for $z, z' \in S^1$, if $\|z - z'\| \leqslant \delta$ then $\|\phi(z) - \phi(z')\| < \min(\frac{1}{2}\varepsilon, \varepsilon_0)$. Let $X \in \Xi$ have $\mathrm{diam}(X) \leqslant \delta$, and let $x \in X$. Let X have ends z, z'. Then $\|\phi(z) - \phi(z')\| < \min(\frac{1}{2}\varepsilon, \varepsilon_0)$, since $\|z - z'\| \leqslant \mathrm{diam}(X)$; and so from (ii) above, $|\psi_X| \subseteq F(\phi(z), \phi(z'))$. In particular, $\psi_X(x) \in F(\phi(z), \phi(z'))$, and by (1)

$$\|\phi(z) - \psi_X(x)\| \leqslant \|\phi(z) - \phi(z')\| \leqslant \tfrac{1}{2}\varepsilon.$$

Moreover, $\|\phi(x) - \phi(z)\| \leqslant \frac{1}{2}\varepsilon$ since $\|x - z\| \leqslant \operatorname{diam}(X) \leqslant \delta$. Hence

$$\|\phi(x) - \psi_X(x)\| \leqslant \|\phi(x) - \phi(z)\| + \|\phi(z) - \psi_X(x)\| \leqslant \varepsilon$$

as required.

We define $\psi: S^1 \to Y \cup bd(\Delta)$ by $\psi(z) = \phi(z)$ if $z \notin Z$ and $\psi(z) = \psi_X(z)$ if $z \in X \in \Xi$. We shall show that ψ satisfies the theorem. Now Ξ is at most countable; let $\Xi = \{X_1, X_2, ...\}$. Define $\psi_i: S^1 \to Y \cup bd(\Sigma)$ for $i = 1, 2, ...,$ by

$$\begin{aligned} \psi_i(z) &= \phi(z) \qquad && (z \in S^1 - X_1 \cup \cdots \cup X_i) \\ &= \psi_{X_j}(z) && (z \in X_j \text{ for some } j \text{ with } 1 \leqslant j \leqslant i). \end{aligned}$$

Clearly $\psi_1, \psi_2, ...,$ are all continuous. Since for all $\delta > 0$ there are only finitely many members $X \in \Xi$ for which $\operatorname{diam}(X) > \delta$, it follows from (3) that the sequence $\psi_1, \psi_2, ...,$ converges uniformly to ψ, and hence that ψ is continuous.

It remains to show that ϕ is homotopic to ψ. The required homotopy is given by

$$\theta(x, t) = t\psi(x) + (1 - t)\,\phi(x) \qquad (x \in S^1, 0 \leqslant t \leqslant 1)$$

as is easily seen. This completes the proof.

The *arc-components* of $Z \subseteq \Sigma$ are the maximal arc-connected subsets of Z; they partition Z. We say $Z \subseteq \Sigma$ is *locally arc-connected* (l.a.c.) if for every $z \in Z$ and every open set $A \subseteq \Sigma$ with $z \in A$ there exists $A' \subseteq A$ with $z \in A'$, open in Σ, such that $Z \cap A'$ is arc-connected.

(11.4) *If Z is connected and l.a.c. then Z is arc-connected.*

Proof. Let A be an arc-component of Z, and let $B = Z - A$. If $z \in \bar{A} \cap \bar{B} \cap Z$, there is an open set $C \subseteq \Sigma$ with $z \in C$ such that $C \cap Z$ is arc-connected; but $A \cap C$, $B \cap C \neq \emptyset$, a contradiction. Thus $\bar{A} \cap \bar{B} \cap Z = \emptyset$, and so $B = \emptyset$ since Z is connected. The result follows.

(11.5) *If $Z \subseteq \Sigma$ is flat and l.a.c., there is an open, flat, l.a.c. set $Y \subseteq \Sigma$ with $Z \subseteq Y$, which may be chosen arc-connected if Z is arc-connected.*

Proof. For each $z \in Z$, choose an open $\Delta_z \subseteq \Sigma$ with $z \in \Delta_z$ such that Δ_z is flat and $Z \cap \Delta_z$ is arc-connected. Since Σ is metrizable, we may choose a metric δ. For each $z \in Z$, choose $\varepsilon(z) > 0$ such that $y \in \Delta_z$ for all $y \in \Sigma$ with $\delta(y, z) \leqslant \varepsilon(z)$. For each $z \in Z$, choose $U_z \subseteq \Delta_z$ with $z \in U_z$, open in Σ and connected, such that $\delta(y, z) < \frac{1}{3}\varepsilon(z)$ for all $y \in U_z$. Let $Y = \bigcup_{z \in Z} U_z$. We claim that Y satisfies the theorem. It suffices to check that Y is flat. Let $\phi: S^1 \to \Sigma$ be continuous with $|\phi| \subseteq Y$. By compactness, we may choose a finite set $K \subseteq S^1$ such that for each component X of $S^1 - K$, there exists $z \in Z$ with $\phi(\bar{X}) \subseteq U_z$. Let $K = \{k_1, ..., k_n\}$, numbered in order in S^1, and define $k_0 = k_n$. Let the components of $S^1 - K$ be X_i $(1 \leqslant i \leqslant n)$ in order, where $\bar{X}_i - X_i = \{k_{i-1}, k_i\} (1 \leqslant i \leqslant n)$. For $1 \leqslant i \leqslant n$, choose $z_i \in Z$ with $\phi(\bar{X}_i) \subseteq U_{z_i}$ and let $z_0 = z_n$. Since $\phi(k_i) \in \phi(\bar{X}_i) \subseteq U_{z_i}$ and U_{z_i} is arc-connected, there is a continuous map $\phi_i: [0, 1] \to U_{z_i}$ with $\phi_i(0) = z_i$ and $\phi_i(1) = \phi(k_i)$. Let $\psi_i = [0, 1] \to \Sigma$ be the concatenation of ϕ_{i-1}, the restriction of ϕ to $\bar{X}_i$, and the reverse of ϕ_i (reparameterizing in the usual way), where $\phi_0 = \phi_n$. We observe that ϕ is homotopic to the concatenation ψ of $\psi_1, \psi_2, ..., \psi_n$. On the other hand, we claim that ψ is homotopic to a curve θ with $|\theta| \subseteq Z$. For let $1 \leqslant i \leqslant n$. Since $|\psi_i| \subseteq U_{z_{i-1}} \cup U_{z_i}$ and $|\psi_i|$ meets both $U_{z_{i-1}}$ and U_{z_i}, it follows that there exists $u \in U_{z_{i-1}} \cap U_{z_i}$. Let z be whichever of z_{i-1}, z_i has $\varepsilon(z)$ the larger. We claim that $U_{z_{i-1}} \cup U_{z_i} \subseteq \Delta_z$. For let $y \in U_{z_{i-1}} \cup U_{z_i}$. Then

$$\min(\delta(y, z_{i-1}), \delta(y, z_i)) < \max(\tfrac{1}{3}\varepsilon(z_{i-1}), \tfrac{1}{3}\varepsilon(z_i)) \leqslant \tfrac{1}{3}\varepsilon(z)$$

$$\delta(u, z_{i-1}) < \tfrac{1}{3}\varepsilon(z_{i-1}) \leqslant \tfrac{1}{3}\varepsilon(z)$$

$$\delta(u, z_i) < \tfrac{1}{3}\varepsilon(z_i) \leqslant \tfrac{1}{3}\varepsilon(z)$$

and so $\max(\delta(y, z_{i-1}), \delta(y, z_i)) < \varepsilon(z)$. Hence $y \in \Delta_z$, as claimed. It follows that

$$|\psi_i| \subseteq U_{z_{i-1}} \cup U_{z_i} \subseteq \Delta_z.$$

Since Δ_z is arc-connected and $\psi_i(0), \psi_i(1) \in Z$, there exists $\theta_i: [0, 1] \to \Sigma$ with $|\theta_i| \subseteq Z \cap \Delta_z$ such that $\theta_i(0) = \psi_i(0)$ and $\theta_i(1) = \psi_i(1)$. Since Δ_z is flat, the concatenation of ψ_i and the reverse of θ_i is null-homotopic, and so ψ is homotopic to the concatenation θ of $\theta_1, \theta_2, ..., \theta_n$. Since $|\theta| \subseteq Z$, θ is null-homotopic, and hence so is ψ and therefore ϕ, as required.

We recall that $Z \subseteq \Sigma$ is *solid* if Z is l.a.c. and closed.

(11.6) *If Z_1, $Z_2 \subseteq \Sigma$ are solid, so is $Z_1 \cup Z_2$.*

Proof. Let $z \in Z_1 \cup Z_2$, and let $A \subseteq \Sigma$ be open with $z \in A$. Let X be the arc-component of $(Z_1 \cup Z_2) \cap A$ which contains z, and let $Y = (Z_1 \cup Z_2) - X$. If $x \in X$, there are open sets $A_1, A_2 \ni x$ such that $A_1, A_2 \subseteq A$ and $Z_i \cap A_i$ is arc-connected, and $Z_i \cap A_i$ is null if $x \notin Z_i$ $(i = 1, 2)$, since Z_1, Z_2 are closed. Hence $Y \cap Z_i \cap A_i = \emptyset$ $(i = 1, 2)$, and so $Y \cap (A_1 \cap A_2) = \emptyset$. Put $A_x = A_1 \cap A_2$. Then A_x is open, and so is

$A' = \bigcup_{x \in X} A_x$. But $A' \cap (Z_1 \cup Z_2) = X$, and so is arc-connected, and $A' \subseteq A$. The result follows.

(11.7) *If $Z \subseteq \Sigma$ is solid and flat, and Σ is connected, there exists $Y \subseteq \Sigma$ with $Z \subseteq Y$ such that Y is solid, flat, and arc-connected.*

Proof. If Z has infinitely many arc-components, there exists $z \in \Sigma$ such that every neighborhood of z meets infinitely many arc-components of Z. Since Z is closed, $z \in Z$; but then z has a neighborhood A such that $A \cap Z$ is arc-connected, a contradiction. Hence Z has only finitely many arc-components. If it has more than one, we may choose a $[0, 1]$-arc $T \subseteq \Sigma$ with its ends in different arc-components of Z and with $|T \cap Z| = 2$. Then $T \cup Z$ is solid and flat by (11.6), and has fewer arc-components than Z. The result follows by repeating this process.

We say $Z \subseteq \Sigma$ is *simply connected* if Z is arc-connected and every O-arc $F \subseteq Z$ bounds a closed disc $\Delta \subseteq Z$. By (11.2), a simply connected set is flat.

(11.8) *If $Z \subseteq \Sigma$ is open, flat, and arc-connected, there is an open, simply connected set $Y \subseteq \Sigma$ with $Z \subseteq Y$.*

Proof. If n is an ordinal, and $Z_i \subseteq \Sigma$ is open, flat, and arc-connected for $i < n$, and $Z_i \subseteq Z_{i'}$ for $i < i' < n$, then $\bigcup_{i<n} Z_i$ is also open, and arc-connected, and flat, by the compactness of O-arcs. Thus, by Zorn's lemma, there exists $Y \subseteq \Sigma$ with $Z \subseteq Y$, open, flat, and arc-connected and maximal with these properties. If $F \subseteq Y$ is an O-arc, there is a closed disc $\Delta \subseteq \Sigma$ bounded by F, since Y is flat. Then $Y \cup \Delta$ is open, arc-connected, and flat, by (11.3); and so $\Delta \subseteq Y$ from the maximality of Y. Thus Y is simply connected, as required.

(11.9) *If Σ is connected and $Z \subseteq \Sigma$ is flat and l.a.c., and either Z is closed or Z is arc-connected, then there is an open simply connected subset $Y \subseteq \Sigma$ with $Z \subseteq Y$.*

Proof. By (11.7) we may assume that Z is arc-connected. By (11.5) we may assume that Z is open. But then the result follows from (11.8).

(11.10) *If Σ is connected and $Z \subseteq \Sigma$ is solid and flat, then either $Z = \Sigma$ or Z is planar.*

Proof. By (11.9) there is an open, simply connected set $Y \subseteq \Sigma$ with $Z \subseteq Y$. If $bd(\Sigma) = \emptyset$ and $Y = \Sigma$, then Σ is a sphere, and the result follows. If $bd(\Sigma) = \emptyset$ and $Y \neq \Sigma$, then Y is an open disc by (4.2), and the result follows since any closed subset of an open disc is included in a closed disc. If $bd(\Sigma) \neq \emptyset$, similar slightly more complicated arguments yield the result, and we omit the details.

Proof of (5.5). The "only if" part of (5.5) is clear. For the "if" part, let Z be as in (5.5), such that every G-normal O-arc included in Z is null-homotopic. Since Z is the closure of the union of some of the regions of G, it follows that Z is solid. The result follows from (11.10) if every continuous map $\phi: S^1 \to Z$ is null-homotopic. Let $\phi: S^1 \to Z$ be continuous. For each loop e of G, choose a point $v_e \in e$, and let G' be the graph with

$$U(G') = U(G)$$
$$V(G') = V(G) \cup \{v_e : e \text{ is a loop of } G\}.$$

Then G' has no loops. For each $e \in E(G')$, since e is not a loop, we may choose a closed disc $\Delta_e \supseteq \bar{e}$, with the ends of e in $bd(\Delta_e)$, with $e \subseteq \Delta_e - bd(\Delta_e)$ and with $\Delta_e \cap U(G') = \bar{e}$; and moreover, we may make these choices so that $\Delta_e \cap \Delta_f \subseteq \bar{e} \cap \bar{f}$ for all distinct $e, f \in E(G')$. For each $e \in E(G')$, either $Z \cap bd(\Delta_e)$ is arc-connected or it consists just of the ends of e; and so if $|\phi| \cap e \neq \varnothing$ then $Z \cap bd(\Delta_e)$ is arc-connected. By applying (11.3) to each Δ_e with $|\phi| \cap e \neq \varnothing$, we deduce that ϕ is homotopic to a map $\psi' : S^1 \to Z$ with $|\psi'|$ G'-normal. It is easy to see that ψ' is homotopic to a map $\psi: S^1 \to Z$ with $|\psi|$ G-normal. By (11.2), ψ is null-homotopic, and hence so is ϕ, as required.

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